Standard Radiation Spectrum of Relativistic Electrons: Beyond the Synchrotron Approximation

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(Received 26 July 2001; published 8 August 2002)

Radiation emitted by an electron in arbitrary, extreme relativistic motion, has been described for the first time in terms of a standard spectrum of nonsynchrotron type. Ultimately, such a nonsynchrotron spectrum is dependant not only on instantaneous trajectory curvature, but also upon its first two time derivatives and helicity, to provide a basic correction to the synchrotron approximation (SA). A strong deviation from SA has been predicted for above GeV electrons in oriented crystals.

DOI: 10.1103/PhysRevLett.89.094801 PACS numbers: 41.60.–m

The radiated power spectrum, emitted by ultrarelativistic electrons undergoing arbitrary motion, was calculated by Schwinger more than 50 years ago [1], in terms of only one local parameter of the trajectory which depends on the instantaneous radius of curvature R. The corresponding spectrum, called synchrotron spectrum, is the same as that emitted by an electron moving along a circular path and is given by a standard function of one variable $\xi = 2\omega/(3\gamma^3 g)$, where $\gamma = (1-\beta^2)^{-1/2} \gg 1$, $\beta = v/c$ is the velocity of an electron, ω is the radiation frequency, $g = \Omega = c/R$ is the local electron's acceleration divided by the velocity of light c: $\mathbf{g} = d\boldsymbol{\beta}_\perp/dt$, where $\boldsymbol{\beta}_\perp = \mathbf{v}_\perp/c$ is the component of velocity transverse to some general direction of motion which we assume to coincide with the z-axis of a coordinate system. For $\gamma \gg 1$, $|\boldsymbol{\beta}_\perp| = \beta_\perp \approx \theta_e \ll 1$, where θ_e is the deflection angle of an electron by the external field, moreover, the z-component of acceleration is negligibly small $\mathbf{g} \approx \mathbf{g}_\perp$ [2] (sect. 14.4) and $\boldsymbol{\beta}\cdot\mathbf{g} \approx 0$.

In recent years synchrotron approximation (SA) has turned out to have important applications in the theory of channeling radiation (CR) (or Kumakhov radiation) [3] in oriented crystals. Such an approximation for CR has been opened up by Kimball and Cue [4] and provides a much simpler description in terms of the well-known classical [1] or quantum [5] synchrotron radiation formulas. Detailed theoretical analysis [6,7] of the recent experimental results [7,8] show that for nearly perfect alignment with respect to the crystal axis and for projectile energies up in the hundreds GeV region, SA explains the basic properties of the radiation spectrum within an accuracy of approximately 10–30%. The validity of SA and possible modifications of the synchrotron spectrum has been studied in Refs. [9,10,11].

Very recently, the authors of Ref. [7] demonstrate that SA has its limitations for multi GeV electrons, and new theory beyond SA is needed. However, up to the present time, no real progress has been achieved to obtain a simple and reasonably precise description of radiation, taking into account deviations of real trajectory from instantaneously pure circular motions. In what follows we shall demonstrate how it could be done successfully in a somewhat simpler manner.

We shall consider in detail the simplest and most important case of an interaction between an electron with an external field, when the initial and the final velocities at $t \to \pm\infty$ are constant and the field [or the electron's acceleration $g(t)$] has only one maximum inside the interaction area.

The time dependence of the electron trajectory with $\gamma \gg 1$ can be expressed via the time dependence of the transverse motion only (see, for instance, [12,13]). The classical Schwinger formula (I.37) in Ref. [1] for spectral distribution of radiation power coming from the time instant t_0, can be written in terms of the transverse motion quantities [12,13], so that

$$P_\omega(t_0) = \frac{e^2\omega}{\pi c \gamma^2} J_\omega(t_0), \tag{1}$$

$$J_\omega(t_0) = \int_0^\infty \left\{1 + \frac{\gamma^2}{2}[\delta\boldsymbol{\beta}_\perp(\tau)]^2\right\} \sin\Delta(\tau)\frac{d\tau}{\tau} - \frac{\pi}{2}, \tag{2}$$

where $\delta\boldsymbol{\beta}_\perp = \boldsymbol{\beta}_\perp(t_+) - \boldsymbol{\beta}_\perp(t_-)$, $t_\pm = t_0 \pm \tau/2$. In further calculations we shall use the quantity $J_\omega(t_0)$, which is proportional to the number of emitted photons. The spectral distribution of total radiation intensity can be obtained by integrating Eq. (1) over t_0 but for our purposes the radiated power is needed.

The phase $\Delta(\tau)$ in Eq. (2) is the odd function of τ and for $\gamma \gg 1$ can be expanded in powers of τ as

$$\begin{aligned}\Delta(\tau) &= \frac{\omega\tau}{2\gamma^2} - \frac{\omega}{2c^2\tau}(\delta\boldsymbol{\rho})^2 + \frac{\omega}{2}\int_{t_-}^{t_+} \boldsymbol{\beta}_\perp^2(\tau')d\tau' \\ &= (3/2)\xi(x + x^3/3 + a_5x^5 + a_7x^7 + \cdots),\end{aligned} \tag{3}$$

where $\delta\boldsymbol{\rho} = \boldsymbol{\rho}(t_+) - \boldsymbol{\rho}(t_-)$, $\boldsymbol{\rho}(t)$ is the transverse coordinate, $x = g\gamma\tau/2$ is Schwinger's dimensionless time variable [1],

$$a_5 = (k'^2 + 3kk'' - 2k^2\mu^2)/(45\gamma^2 k^4) \tag{4}$$

$$= (A + 3B)/45, \tag{5}$$

where $A = \dot{\mathbf{g}}^2/\gamma^2 g^4$, $B = \mathbf{g}\ddot{\mathbf{g}}/\gamma^2 g^4$, $\dot{\mathbf{g}}$ and $\ddot{\mathbf{g}}$ are the time derivatives of acceleration taken at t_0, $k = 1/R$ is the trajectory's curvature, $k' \equiv dk/ds$, $k'' \equiv d^2k/ds^2$, $ds = vdt$, and μ is the instantaneous helicity.

The expansion of the factor in front of sin in (2) has the form of

$$1 + \gamma^2(\delta\boldsymbol{\beta}_\perp)^2/2 = 1 + 2x^2 + b_4x^4 + b_6x^6 + \ldots, \tag{6}$$

with $b_4 = 2B/3$.

The important parameter for further consideration is the "non-dipole parameter" $\nu_{nd} = \theta_e\gamma \approx \beta_\perp\gamma$. When $\nu_{nd} \ll 1$ the radiation formulas simplify to dipole approximation [13]. In the opposite limit $\nu_{nd} \gg 1$ the SA is valid.

When $\gamma \to \infty$, i.e., $\nu_{nd} \gg 1 \sim \gamma^{1/2}$, the coefficients a_i and b_i vanish. Namely, a_5 and $b_4 \sim \gamma^{-1}$, a_7 and $b_6 \sim \gamma^{-2}$, etc. In the synchrotron limit all $a_i = b_i = 0$ and Eqs. (2)–(6) give the well-known synchrotron result [1,14]

$$J_\xi(t_0) = \frac{1}{\sqrt{3}}\left[2K_{2/3}(\xi) - \int_\xi^\infty K_{1/3}(\lambda)d\lambda\right], \tag{7}$$

where K_ν are the McDonald functions. The above defined Schwinger's frequency variable ξ can be rewritten with account of the quantum recoil [5,15,16] as

$$\xi(t_0) = (2/3)[u/(1-u)]/\chi(t_0), \tag{8}$$

where $u = \hbar\omega/E$, $E = \gamma mc^2$, and $\chi(t_0) = \hbar\gamma F(t_0)/(m^2c^3)$ is the invariant field parameter depending on the local force $F(t_0)$.

Instead of using an instantaneously circular motion, let us approximate an arbitrary trajectory by some motion which has constant transverse velocities $\boldsymbol{\beta}_{\perp 2}$ and $\boldsymbol{\beta}_{\perp 1}$ when $t \to \pm\infty$ and the acceleration achieves its maximum value at $t = 0$. The simplest motion of this type we have found is the following:

$$\boldsymbol{\beta}_\perp(t) = \mathbf{b_0} + \mathbf{b}\tanh(t/T), \tag{9}$$

where $\mathbf{b}_0 = (\boldsymbol{\beta}_{\perp 1} + \boldsymbol{\beta}_{\perp 2})/2$, $\mathbf{b} = (\boldsymbol{\beta}_{\perp 2} - \boldsymbol{\beta}_{\perp 1})/2$, and the acceleration is proportional to $\mathbf{g}(t) \sim \cosh^{-2}(t/T)$, where T is the typical interaction time of an electron with an external field.

We shall call Eq. (9) a standard th-trajectory. The integrand of Eq. (2) can be calculated for th-trajectory analytically. The phase (3) takes the form

$$\Delta(z) = \frac{3\xi}{2\,C^2}\left[\nu z(1+\nu^2) - \frac{\nu^3}{4z}\ln^2\!\left(\frac{1+\eta\tanh z}{1-\eta\tanh z}\right) - \frac{\nu^3\sinh z\cosh z}{C^2+\sinh^2 z}\right], \tag{10}$$

where $\nu = |\mathbf{b}|\gamma$, $z = \tau/2T$, $C = \cosh(t_0/T)$, and $\eta = \tanh(t_0/T)$. Equation (10) contains two papameters ν and C apart from the synchrotron parameter ξ.

Our aim is to apply Eq. (9) for arbitrary motion in such a way that the expansion (3) of the real trajectory would give the same coefficients a_5 and b_4 as that calculated for the th-trajectory. This can be done since having known two parameters A and B for the real trajectory in Eq. (5) one can find two parameters ν and C for the th-spectrum in Eq. (10). Since the main amount of radiation comes from the vicinity of the point $t_0 = 0$ we shall approximate the arbitrary trajectory by Eq. (9) taking the values of C, ν, and η in Eq. (10) at $t_0 = 0$ and calculating the actual values of a_5 at the same point (i.e., at $t_0 = 0$).

Substituting Eq. (10) with $C = 1$ and $\eta = 0$ into Eq. (9), we obtain a nonsynchrotron spectrum (NSS) for arbitrary trajectory, depending on two parameters ξ and ν:

$$J_\xi(\nu, t_0) = \int_0^\infty (1 + 2\nu^2\tanh^2 z) \times \sin\left\{\frac{3}{2}\xi\nu[(1+\nu^2)z - \nu^2\tanh z]\right\}\frac{dz}{z} - \frac{\pi}{2}, \tag{11}$$

where

$$\nu = (2/15)^{1/2}\ |\,a_5\,|^{-1/2}. \tag{12}$$

The expansion of the phase in Eq. (11) gives $a_5 = -2/(15\nu^2)$, therefore having known a_5 for real traectory, calculated at $t_0 = 0$ one can find ν from Eq. (12) and calculate the spectrum from Eq. (11). The t_0-dependence of NSS [Eq. (11)] is due to $\xi = \xi(t_0)$. The coefficient a_5 for NSS will be exactly the same as that for the real one at the point where the main amount of radiation comes from.

Equation (11) represents the main result of our paper. As we shall see below such an approximation gives reasonable quantitative results.

Now we demonstrate its validity by comparison with some exact analytical solutions. Such a solution exists for scattering of relativistic electron by the Coulomb-like axial field of charged string $U(\rho) = -\alpha/\rho$ [see Eq. (72) in Ref. [3]], where for axial electron quasichanneling $\alpha \approx 3Ze^2a_{TF}/(2d)$, Z is the target atomic number, a_{TF} is the Thomas-Fermi screening parameter, and d is the distance between atoms in the atomic string. In this case the spectrum is determined by two parameters: the trajectory eccentricity $\varepsilon \geq 1$ and the Coulomb nondipole parameter $\nu_{nd} \equiv \nu_0$ [3,12]. We consider here only the unbound transverse motion, then Eqs. (4) and (11) for Coulomb-like axial field give $\nu = \nu_0\, 3^{1/2}\,(4\varepsilon^2 - 3\varepsilon - 1)^{-1/2}$.

Figure 1 demonstrates a good degree of accuracy given by the present theory. The quantities plotted in Fig. 1 show the intensity spectrum, i.e., the power $uJ_u(\nu, t_0)$ integrated over all t_0 (here $u \equiv \hbar\omega/E$) for $\nu_0 = 1$ and $\varepsilon = 3$ (therefore, $\nu = 0.34$). This example corresponds to 2.5 GeV electrons with transverse energies $dE_\perp/Ze^2 = 1$ in Si⟨110⟩ crystal. It should be noted that SA (the dotted

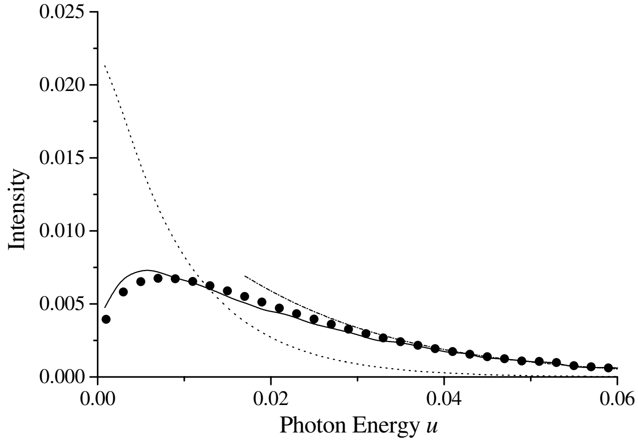

FIG. 1. Coulomb-like intensity spectrum for $\nu_0 = 1$ and $\varepsilon = 3$. Solid curve is the exact solution, black symbols-NSS-approximation (10), dotted curve-SA. Dashed curve is the asymptotics (13), $u \equiv \hbar\omega/E$.

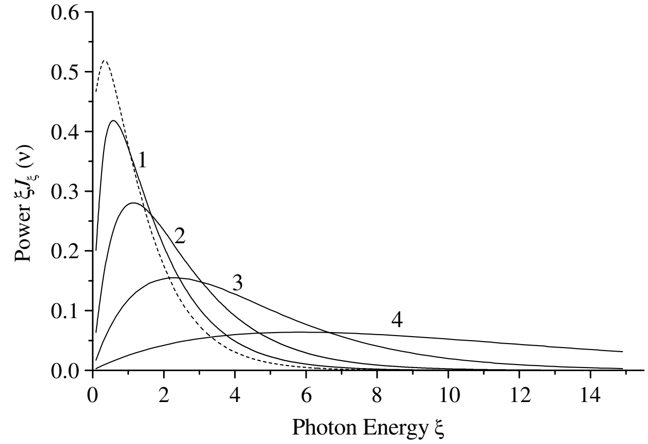

FIG. 2. The standard nonsynchrotron power spectrum (NSS) $\xi J_\xi(\nu, \xi)$ as a function of ξ variable with $J_\xi(\nu, \xi)$ given by (11). Dashed line is the synchrotron spectrum (7) (times ξ). Curves 1, 2, 3, and 4 correspond to $\nu = 1$, 0.5, 0.25, and 0.1, respectively.

curve) is not valid for all the frequencies. Other calculations for different ν_0 and ε parameters (i.e., for different electron energies and absolute values of the external force) indicate a good degree of accuracy even in the dipole limit when $\nu_0 \ll 1$.

The phase of Eq. (11) has a saddle point lying on the imaginary axis of the time variable z: $\text{Im} z_0 \equiv y_0$, where $\tan y_0 = 1/\nu$. For large frequencies $\xi > 1$ the asymptotics for NSS is

$$J_\xi(\nu) \to \frac{1}{2y_0}\left[\frac{2\pi}{3\xi(1+\nu^2)}\right]^{1/2} \exp\left\{\frac{3}{2}\xi\nu[\nu - y_0(1+\nu^2)]\right\}. \tag{13}$$

For $\nu \gg 1$, we get $y_0 \approx 1/\nu - \nu^{-3}/3$ and (13) gives the well-known synchrotron asymptotics [1] $J_\xi^{SA} \to (2\pi/3\xi)^{1/2} \exp(-\xi)/2$.

In the limit of low frequencies $\omega \to 0$, a strong suppression of radiation arises comparing with SA. Thus, the spectral density of radiated power in Eq. (11) goes to constant value $J_\xi \to \pi\nu^2$ when $\omega \to 0$, whereas the synchrotron formula (7) diverges as $\sim\omega^{-2/3}$.

NSS for different values of $\nu \leq 1$ is plotted in Fig. 2, where we have drawn the power spectrum $\xi J_\xi(\nu)$. It is clear from Fig. 2 that the NSS is such that for $\nu > 1$ it has a shape close to the synchrotron approximation except at the very low $\xi \ll 1$ and very high $\xi \gg 1$ frequencies. Therefore, the condition of validity of the SA for the most of the classical spectrum is $\nu > 1$.

The effect of strong deviation from SA at $\nu < 1$ (which condition coincides with the dipole approximation condition) can be understood as following. The critical synchrotron frequency is $\omega_c \approx g\gamma^3$, whereas the typical dipole radiation frequency is $\omega_{\max} \approx \omega_0\gamma^2$ with $\omega_0 \approx 1/T$, where $1/T = g\gamma/\nu$ (for $t_0 = 0$) and $\omega_c/\omega_{\max} \approx \nu$. Therefore, in the dipole limit (when $\nu \ll 1$) the typical radiation frequencies $\omega \sim \omega_{\max}$ are much higher than the synchrotron critical frequency ω_c. For $\nu \gg 1$, $\omega_{\max}$ does not describe any more the typical radiation frequencies due to the contribution of higher harmonics and the spectrum becomes of synhrotron nature. SA assumes the constant field approximation for radiation, i.e., that an electron is always moving in the constant field and radiation spectrum is not sensitive to how the initial and final velocities are. If the deflection angle of an electron by the external field θ_e is smaller than the typical radiation angle $1/\gamma$, a strong correlation between the two parts of the trajectory with constant velocities affect the radiation spectrum which becomes relatively broad and is characterized by two parameters, one of which (ξ) depends on the field and the second one (ν) contains the information about the initial and the final velocities. In the opposite limit $\theta_e > 1/\gamma$, such correlation is weak and the radiation spectrum is completely defined by the local field, thus the spectrum is characterized by only one parameter ξ.

At high electron energies when the invariant field parameter $\chi \geq 1$ the influence of spin on the radiation becomes important in the hard frequency region $\hbar\omega \sim E$. The corresponding modification of the radiation spectrum is very simple [15], namely, we must replace the factor ν^2 in front of $\tanh^2(z)$ in Eq. (11) by $\nu^2(1 + p/2)$, where $p = u^2/(1-u)$ is the spin term. The corresponding asymptotics (13) should be multiplied by $1 + p$. As we shall demonstrate below the quantum spectrum at high electron energies can greatly differ from SA even if $\nu > 1$.

The important practical application of the foregoing analysis is the study of the CR in the oriented crystals for a sufficiently realistic potential. In the calculations presented below we have used the Doyle-Turner atomic potential. The power spectra $uJ_u(\nu)$ of 4 TeV quasichanneled electrons radiating from the same point $\rho = a_{TF}$ near the $\langle 111 \rangle$ axis in Si ($\chi = 15.6$) are shown in Fig. 3. In the SA

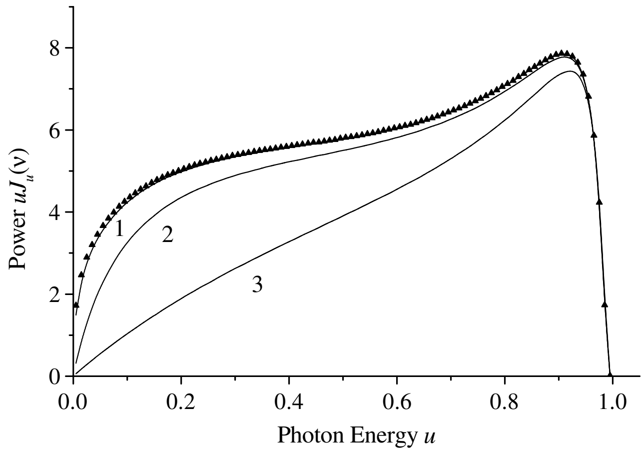

FIG. 3. The radiation power spectrum coming from the same distance $\rho = a_{TF}$ to $\langle 111 \rangle$ axis in Si, for 4-TeV quasichanneled electrons with different transverse energies $\varepsilon_\perp = dE_\perp/Ze^2$. Curves 1, 2, and 3 correspond to $\varepsilon_\perp = 1$, 20, and 120, respectively. Black symbols represent SA. The values of angular momenta are equal to the maximum possible ones for given transverse energy.

(black symbols) the electrons with different incident angles $\theta_{in} > \theta_L$ give the same spectra, where $\theta_L = (4Ze^2/dE)^{1/2}$ is the critical channeling angle [17]. On the other hand, the more precise calculation according to Eq. (11) with account for quantum effects of recoil and spin give a strong dependence of the radiation on θ_{in}. Curves 1, 2, and 3 correspond to the incident angles $\theta_{\text{in}} \approx \theta_L$, $3\theta_L$, and $8\theta_L$ ($\nu = 17.2$, 4.7, and 1.9). It follows from Fig. 3 that for multi-GeV electrons SA is a good approximation for small incident angles $\theta_{in} \approx \theta_L$ [9,18] while for larger incident angles a strong suppression of radiation occurs in the whole spectral region except for $\hbar\omega \sim E$. We, therefore, predict a strong suppression of radiation for above hundreds GeV electrons in about the whole spectrum compared with that for SA for incident angles $\theta_{in} > \theta_L$.

In summary, by introducing a model trajectory, we have derived a simple expression of nonsynchrotron spectrum (NSS) which coincides with SA at $\nu \gg 1$ while approaches to the dipole radiation at $\nu < 1$. NSS may also have not only important applications to channeling, but also astrophysical radiation [19,20] as well as radiation in an accelerator. For example, beamstrahlung in a linear collider has been considered within SA [21], which is clearly inadequate especially at very high energies (cf. Fig. 3). By using NSS, one can take into account the effect beyond SA easily in a simulation. Finally, we mention that NSS is also useful to consider pair production in strong fields where the semi-classical calculations based on SA with the use of the crossing symmetry has turned out to be inapplicable [22].

M. K. thanks the late professor Jens Lindhard for his kind hospitality during his stay in Aarhus University in the midd 1990's, and for bringing to M. K.'s attention the problems discussed in this paper. We also wish to thank Luke Gerrard for assistance. This research was supported by Japan Cultural Association.

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