

## Stable Quantum Resonances in Atom Optics

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A theory for stabilization of quantum resonances by a mechanism similar to one leading to classical resonances in nonlinear systems is presented. It explains recent surprising experimental results, obtained for cold cesium atoms when driven in the presence of gravity, and leads to further predictions. The theory makes use of invariance properties of the system allowing for separation into independent kicked rotor problems. The analysis relies on a fictitious classical limit where the small parameter is *not* Planck's constant, but rather the detuning from the frequency that is resonant in the absence of gravity.

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Resonances are among the most typical manifestations of the quantum mechanical behavior. They are characterized by a precise relation between the parameters of the system and an external driving; for instance, resonant excitation of an atom requires that the difference between two levels be equal to the frequency of the driving field times Planck's constant. Classical resonances of nonlinear systems are more robust. Regions of appreciable size in phase space may respond in a resonant way, while other regions are chaotic. In recent experiments [1–3], it was found that a quantum system that is *far* from its semiclassical limit exhibits resonant behavior very similar in nature to the one found for classical nonlinear systems: it is robust, and the high precision that is characteristic of the usual quantum resonances is not required. It is observed for various values of the parameters and of the initial conditions. In this Letter, the experimental results are explained and further predictions are made. Details are given in [4]. The theory reveals how resonances, similar to those characteristic of classical systems, may arise in a quantum system, far from the semiclassical limit. The theoretical model of the experiment is a variant of the kicked rotor, which is a standard system used in the investigation of classical Hamiltonian chaos and its manifestations in quantum mechanical systems [5,6]. In experimental realizations of the model, laser-cooled atoms are driven by application of a standing electromagnetic wave. The frequency of the wave is slightly detuned from resonance, so a dipole moment is induced in the atom. This moment couples with the driving field, giving rise to a net force on the center of mass of the atoms, proportional to the square of the electric field [7,8]. As the wave is periodic in space, the atom is thus subjected to a periodic potential. The wave is turned on and off periodically in time, resulting in a sequence of short pulses. A realization of a periodically kicked particle is then obtained. For atoms driven in the horizontal direction, effects of gravity are negligible, and localization in momentum is observed [9], as theoretically expected (see below).

In the experiments that provide the subject of the theoretical analysis of the present Letter, laser-cooled cesium atoms were driven in the vertical direction, and gravity was found to produce new remarkable effects. Localization was destroyed; furthermore, in the vicinity of the frequencies that are resonant in the absence of gravity, a new type of ballistic spread in momentum was experimentally observed [1–3]. A fraction of the atoms were steadily accelerated, at a rate which was faster or slower than the gravitational acceleration depending on what side of the resonance the driving frequency was. Such atoms were exempt from the diffusive spread that took place for the other atoms, and their acceleration depended on the difference between the driving and the resonant frequencies. The dynamics of the atoms that are falling as a result of gravity and are kicked by the external field is modeled by the time-dependent Hamiltonian:

$$\hat{H}(t') = \frac{\hat{P}^2}{2} - \frac{\eta}{\tau} \hat{X} + k \cos(\hat{X}) \sum_{t=-\infty}^{+\infty} \delta(t' - t\tau), \quad (1)$$

where  $t'$  is the continuous time variable, the integer variable  $t$  counts the kicks,  $\hat{P}$ ,  $\hat{X}$  are the momentum and the position operator, respectively. Units are chosen so that the mass of the atoms is 1, the Planck's constant is 1, and the spatial period of the kicks is  $2\pi$ . The dimensionless parameters  $k$ ,  $\eta$ ,  $\tau$  fully characterize the dynamics. They are expressed in terms of the physical parameters as follows:  $k = \kappa/\hbar$ ,  $\tau = \hbar T G^2/M$ ,  $\eta = MgT/(\hbar G)$ , where  $M$ ,  $T$ ,  $\kappa$ ,  $g$  are the mass of the atoms, the kicking period, the kick strength, and the gravitational acceleration, respectively, and  $2\pi/G$  is the spatial period of the kicks. The positive  $x$  direction is that of the gravitational acceleration. *Throughout the following, time is a discrete variable, given by the kick counter  $t$ .* It is expedient to measure the momentum in the free falling frame, notably to replace  $\hat{P} - \frac{\eta}{\tau} t'$  by  $\hat{P}$ , resulting in the time-dependent Hamiltonian :

$$\hat{H}(t') = \frac{1}{2}(\hat{P} + \frac{\eta}{\tau}t')^2 + k \cos(\hat{X}) \sum_{t=-\infty}^{+\infty} \delta(t' - t\tau). \quad (2)$$

This Hamiltonian is related to (1) by the gauge transformation  $e^{i\eta\hat{X}t'/\tau}$ . The main advantage of (2) compared to (1) is that it depends on  $\hat{X}$  only via  $\cos(\hat{X})$ . Consequently, the evolution mixes only momenta which differ by integers: hence, quasimomentum is conserved. In the absence of gravity ( $\eta = 0$ ), (2) reduces to the Hamiltonian of the kicked rotor (classically resulting in the standard map) with stochasticity parameter  $K = k\tau$ . For  $K > K_c \approx 0.9716$ , diffusion in momentum takes place. For values of  $K$  near to integer multiples of  $2\pi$ , accelerator modes are found [6]. These are stable phase-space islands that travel ballistically, resulting in linear, rather than diffusive, growth of momentum with time. The quantum mechanical study of the systems described by (2) starts with decomposing the momentum as  $p = n + \beta$ , where  $n, \beta$  denote the integer and the fractional parts of the momentum  $p$ , respectively. Since the quasimomentum  $\beta$  is conserved only  $n$  varies in the course of the dynamics. The evolution is equivalent to that of a superposition of independent kicked rotors, each characterized by a different value of  $\beta$ . Such a rotor will be called the  $\beta$  rotor, and the one step evolution operator is

$$\hat{U}_\beta(t) = e^{-ik \cos(\hat{\theta})} e^{-i(\tau/2)(\hat{N} + \beta + \eta t + \eta/2)^2}. \quad (3)$$

where  $\theta = x \bmod(2\pi)$  and the momentum operator is  $\hat{N} = -i \frac{d}{d\theta}$ . For  $\beta = 0$  and  $\eta = 0$  this is the usual kicked rotor. The classical diffusion is then suppressed by a mechanism that is similar to Anderson localization in disordered solids [5,10]. For  $\tau = 2\pi l/m$ , where  $l$  and  $m$  are integers, the eigenstates of (3) are extended in momentum and ballistic growth of momentum takes place in most cases. This is the quantum resonance [11]. The Talbot length corresponds to  $4\pi$ . For fixed  $\beta \neq 0$  and for typical irrational  $\tau/2\pi$ , the classical diffusion is again suppressed, as was observed in experiments on laser-cooled sodium and cesium atoms [9]. Quantum resonances occur only at special values of  $\beta$ ; hence, their experimental observation is difficult [12], since it is impossible to prepare all atoms with the same quasimomentum  $\beta$ . Averaging over  $\beta$  results in linear (rather than quadratic) growth of the squared momentum [4]. This demonstrates the high sensitivity of the quantum resonances to fine experimental details.

In the presence of gravity,  $\eta \neq 0$ , the evolution operator (3) is time dependent. The localization in momentum is destroyed, as it was experimentally observed [1–3]. While this is expected for time-dependent random potentials, the present time dependence is just quasiperiodic, so a deeper analysis is required. However, the most surprising experimental result was that an appreciable fraction of atoms were found to accelerate (in the free falling frame) for various values of the experimental parameters, for values of  $\tau$  in intervals of appreciable size around integer multi-

ples of  $2\pi$ . Here is a quantum resonance that is robust, in contrast to the usual quantum resonances. It is reminiscent of classical nonlinear resonances, namely, the accelerator modes of the standard map; however, *it has no counterpart in the classical limit of (2)*. In what follows, a theoretical explanation of this effect is presented [4]. We consider the case when  $\tau$  is close to a resonant value  $2\pi l$  ( $l > 0$  integer), and the kicking strength  $k$  is large. We, hence, write  $\tau = 2\pi l + \epsilon$ ,  $k = \tilde{k}/|\epsilon|$  with  $\epsilon$  small. Noting  $e^{-i\pi l n^2} = e^{-i\pi l n}$ , (3) takes the following form (apart from an irrelevant phase factor):

$$\hat{U}_\beta(t) = e^{-(i/|\epsilon|)\tilde{k} \cos \hat{\theta}} e^{-(i/|\epsilon|)\hat{H}_\beta(\hat{I}, t)}, \quad (4)$$

where

$$\hat{I} = |\epsilon| \hat{N} = -i|\epsilon| \frac{d}{d\theta},$$

$$\hat{H}_\beta(\hat{I}, t) = \frac{1}{2} \text{sign}(\epsilon) \hat{I}^2 + \hat{I} \left[ \pi l + \tau \left( \beta + t\eta + \frac{\eta}{2} \right) \right]. \quad (5)$$

If  $|\epsilon|$  is assigned the role of Planck's constant, then (4) and (5) is the formal quantization of either of the following classical (time-dependent) maps :

$$I_{t+1} = I_t + \tilde{k} \sin(\theta_{t+1}),$$

$$\theta_{t+1} = \theta_t \pm I_t + \pi l + \tau(\beta + t\eta + \eta/2) \bmod(2\pi), \quad (6)$$

where  $\pm$  has to be chosen according to the sign of  $\epsilon$ . The small  $|\epsilon|$  asymptotics of the quantum  $\beta$  rotor is thus equivalent to a quasiclassical approximation based on the “classical” dynamics (6). We emphasize that classical here is not related to the  $\hbar \rightarrow 0$  limit but to the limit  $\epsilon \rightarrow 0$  instead. The two limits are actually incompatible with each other except possibly when  $l = 0$ . For the sake of clarity, the term “ $\epsilon$  classical” will be used in the following. Changing the variable to  $J_t = I_t \pm \pi l \pm \tau(\beta + \eta t + \eta/2)$  removes the explicit time dependence of the maps (6), yielding:

$$J_{t+1} = J_t + \tilde{k} \sin(\theta_{t+1}) \pm \tau\eta, \quad \theta_{t+1} = \theta_t \pm J_t. \quad (7)$$

If  $J, \theta$  are taken  $\bmod(2\pi)$ , then (7) define maps of the 2-torus in itself. Let  $J_0, \theta_0$  be a period  $\mathbf{p}$ -fixed point of either of the toral maps thus defined. Then iteration of (7) yields, at  $t = \mathbf{p}$ :

$$J_{\mathbf{p}} = J_0 + 2\pi\mathbf{j}, \quad \theta_{\mathbf{p}} = \theta_0 + 2\pi m, \quad (8)$$

for some integers  $\mathbf{j}, m$ . In terms of the original dynamics (6), this yields a family of orbits such that, for all integers  $t$ ,

$$\theta_{\mathbf{p}t} = \theta_0 = \vartheta_0 \bmod(2\pi), \quad I_{\mathbf{p}t} = I_0 + a\mathbf{p}t, \quad (9)$$

where  $a = \mp \tau\eta + 2\pi\mathbf{j}/\mathbf{p}$  and

$$I_0 = J_0 \mp \pi l \mp \tau(\beta + \eta/2) + 2\pi m', \quad (10)$$

with  $m'$  any integer. Thus, primitive periodic orbits of the toral maps (7) correspond to families of *accelerator orbits* of the dynamics (6), marked by linear average growth of

momentum with time. If the periodic orbits are stable, then the accelerator orbits are surrounded by islands of positive measure in phase space, also leading to ballistic (linear) average growth of momentum in time. These are named *accelerator modes*. They are characterized by the integer couple  $(\mathbf{p}, \mathbf{j})$  formed by the *order*  $\mathbf{p}$  and by the *jumping index*  $\mathbf{j}$ . Modes of order 1 correspond to fixed points of (7). They are given on the 2-torus by  $J_0 = 0$ ,  $\theta_0 = \theta_j$ , where

$$\sin(\theta_j) = (2\pi\mathbf{j} \mp \tau\eta)/\tilde{k}, \quad (11)$$

and  $\mathbf{j}$  is any integer such that the right-hand side is not larger than 1 in absolute value. A linear stability analysis shows that, for any integer  $\mathbf{j}$ , each map (7) has exactly one stable period-1 fixed point on the 2-torus, given by (11) if, and only if,

$$|2\pi\mathbf{j} \mp \tau\eta| < \tilde{k} < \sqrt{16 + (2\pi\mathbf{j} \mp \tau\eta)^2}. \quad (12)$$

When  $\tilde{k}$  increases beyond the right-hand limit, such fixed points turn unstable and bifurcations occur. At small values of  $\tilde{k}$ , higher-order accelerator modes appear, associated with higher-period primary orbits. Examples of stable periodic orbits near  $\tau = 2\pi$  are presented in Fig. 1. Initial physical momenta  $n_0 = |\epsilon|^{-1}I_0$  for  $\epsilon$ -classical accelerator modes are obtained from (10) for any  $0 \leq \beta < 1$ . If the stable islands associated with  $\epsilon$ -classical accelerator modes have a large area compared to  $|\epsilon|$ , then they may trap some of the rotor's wave packet and give rise to *quantum accelerator modes* traveling in physical momentum space with speed  $\sim a/|\epsilon| = -\tau\eta/\epsilon + 2\pi\mathbf{j}/(\mathbf{p}|\epsilon|)$ . In order that such modes may be observed, the phase-space distribution associated with the initial rotor state must significantly overlap the islands. This picture is confirmed by numerical simulations. Figure 2 shows the quantum phase-space evolution of a  $\beta$  rotor with  $\beta = 0.2188$  started in the coherent state centered at the position of a (1,0)-accelerator mode. The Husimi functions computed at subsequent times closely follow the motion of the  $\epsilon$ -classical mode. In the  $\epsilon$ -semiclassical regime, accelerator modes are expected to

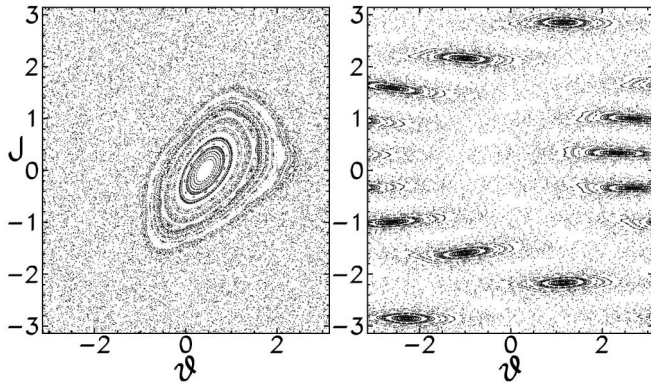


FIG. 1. Phase portraits for the map (7) on the 2-torus, with  $\eta/\tau = 0.01579$ . Left: stable (1,0) fixed point  $J_0 = 0$ ,  $\theta_0 = 0.42$ , for  $\tau = 5.86$ ,  $\tilde{k} = 1.329$ . Right: stable (10,1) periodic orbit at  $\tau = 6.31$ ,  $\tilde{k} = 0.067$ .

decay exponentially in time due to quantum tunneling out of the classical islands, with decay rate  $\gamma_\epsilon \propto \exp[-(\text{const}/|\epsilon|)]$ . This decay was also numerically confirmed [4].

In the experiments described in [1–3], the initial state of the falling atoms is satisfactorily described by an incoherent Gaussian mixture of momentum eigenstates centered at  $p = 0$ , with rms deviation  $\approx 2.55$  in our units. This is equivalent to a statistical ensemble of  $\beta$  rotors. The above theory describes this situation for small  $\epsilon$ , where the  $\epsilon$ -classical approximation holds. From (9) it is found that the  $\epsilon$ -classical  $(\mathbf{p}, \mathbf{j})$  mode started at  $t = 0$  with  $I_0 = n_0|\epsilon|$  is located at time  $t$  at the momentum:

$$n \simeq n_0 - t\tau\eta/\epsilon + 2\pi t\mathbf{j}/(\mathbf{p}|\epsilon|). \quad (13)$$

In Fig. 3, this prediction is compared to the results of numerical simulations in the vicinity of  $\tau = 2\pi$  for  $k = 0.8\pi$  and  $\eta/\tau = 0.01579$  (that is the value of the gravitational acceleration in our units). The time is  $t = 60$  kicks. The initial state is an incoherent mixture of 50 momentum states, distributed as reported in experimental papers. Lines are as predicted by (13). Good agreement is found. The mode (1,0) was identified experimentally (as can be seen on comparing Fig. 3 to Fig. 2 of [1]; also recall that our positive direction is that of the gravitational acceleration). Further higher-order modes are predicted in the present work. Longer time is required for their resolution, because they move slower, yet they may be eventually resolved (see the inset of Fig. 3). Similar behavior is predicted for  $\tau \approx 4\pi$  (as described in [4]), and for higher multiples of  $2\pi$  as well. We finally remark that (i) quantization of the  $\beta$  rotors momentum enhances quantum modes at discrete values of  $\beta$ , given by (10) with  $I_0$  an integer multiple of

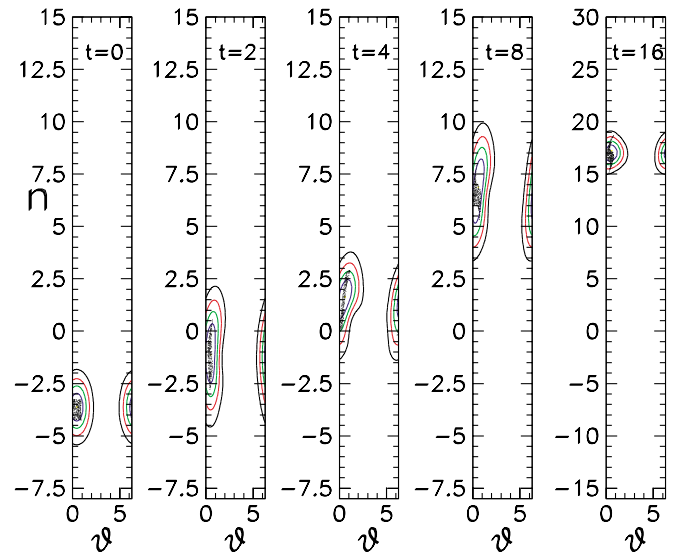


FIG. 2 (color online). Contour plots of Husimi functions at times  $t = 0, 2, 4, 8$ , and  $16$  for a  $\beta$  rotor initially prepared in the coherent state centered at the  $\epsilon$ -classical (1,0) accelerator mode of Fig. 1 (left). The black spots in the centers of the contours are an ensemble of classical points of size  $\hbar = 1$ , evolving according to the  $\epsilon$ -classical dynamics (6).

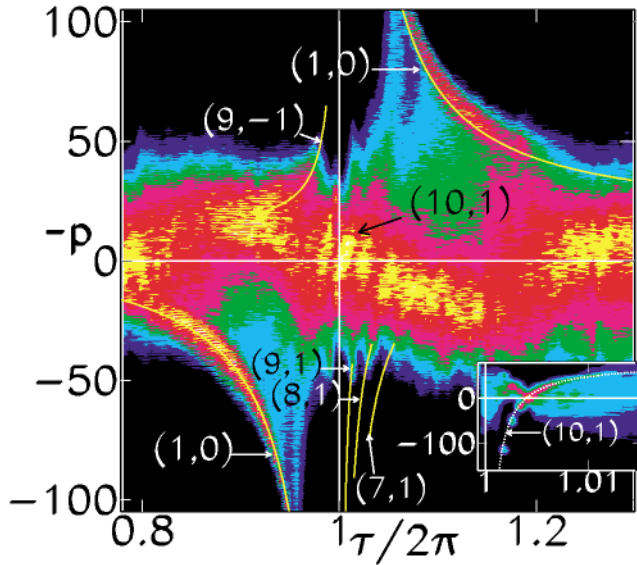


FIG. 3 (color). Momentum distribution in the falling frame at  $t = 60$  kicks for different values of the kicking period  $\tau$  near  $\tau = 2\pi$ . Note the negative sign of  $p$ . Yellow corresponds to highest probability. The initial state is a mixture of plane waves sampled from a Gaussian distribution of momenta. Full lines are the theoretical curves (13), with orders and jumping indices as indicated by the arrows. The inset shows data at  $t = 400$ .

$\epsilon$ . This results in preferred values for physical momentum  $p = n + \beta$ , spaced by  $\approx 2\pi l/\tau$ . This was experimentally detected; (ii) the accelerator modes of the  $\beta$  rotors result in acceleration of the particle also in the case when the initial state is a coherent superposition of momentum eigenstates[4].

The key steps in the theory presented in this work are as follows: (i) transformation to the free falling frame leading to the quasimomentum conserving Hamiltonian (2); (ii) introduction of the  $\beta$ -rotors and approximation of their motion by the one generated by the  $\epsilon$ -classical Hamiltonian (5) near  $\tau = 2\pi l$ ; (iii)  $\epsilon$ -classical accelerator modes (10) dominate the dynamics near  $\tau = 2\pi l$ .

What stabilizes these accelerator modes? On top of the long wavelength behavior related to the resonance  $\tau = 2\pi l$ , there is a short wavelength behavior. Its dynamics is dominated by the fictitious classical mechanics termed  $\epsilon$  classical. This is reminiscent of the “caustics without rays” [13]. The fact that the intermediate-time dynamics is dominated by a discrete set of modes, which exponentially decay in time, is resemblant of the Wannier-Stark problem of a Bloch particle in a constant field. How far this analogy carries is, in our opinion, an interesting theoretical question. Other theoretical questions well worth further analysis are about the nature of the asymptotic (in time) behavior and of the  $\epsilon$ -semiclassical systematic expansion. On the experimental side, it would be interesting to classify the various accelerator modes, namely, to perform “accelerator mode spectroscopy”; and also to experimentally

explore differences in the quantum dynamics between coherent and incoherent superpositions in the initial state of the atoms. This may be of importance for atomic interferometry and for quantum computation.

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- [1] M.K. Oberthaler, R.M. Godun, M.B. d’Arcy, G.S. Summy, and K. Burnett, *Phys. Rev. Lett.* **83**, 4447 (1999).
- [2] R.M. Godun, M.B. d’Arcy, M.K. Oberthaler, G.S. Summy, and K. Burnett, *Phys. Rev. A* **62**, 013411 (2000).
- [3] M.B. d’Arcy, R.M. Godun, M.K. Oberthaler, G.S. Summy, K. Burnett, and S.A. Gardiner, *Phys. Rev. E* **64**, 056233 (2001); M.B. d’Arcy, R.M. Godun, M.K. Oberthaler, D. Cassettari, and G.S. Summy, *Phys. Rev. Lett.* **87**, 074102 (2001).
- [4] S. Fishman, I. Guarneri, and L. Rebuzzini, *nlin.CD/0202047* [*J. Stat. Phys.* (to be published)].
- [5] For reviews, see, e.g., F.M. Izrailev, *Phys. Rep.* **196**, 299 (1990); S. Fishman, in *Proceedings of the International School of Physics Enrico Fermi: Varenna Course CXIX*, edited by G. Casati, I. Guarneri, and U. Smilansky (North-Holland, Amsterdam, 1993), p. 187.
- [6] A.J. Lichtenberg and M.A. Lieberman, *Regular and Chaotic Dynamics* (Springer-Verlag, New York, 1992).
- [7] R. Graham, M. Schlautmann, and P. Zoller, *Phys. Rev. A* **45**, R19 (1992).
- [8] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Atom-Photon Interactions, Basic Processes and Application* (Wiley, New York, 1992), p. 456.
- [9] D.A. Steck, V. Milner, W.H. Oskay, and M.G. Raizen, *Phys. Rev. E* **62**, 3461 (2000); F.L. Moore, J.C. Robinson, C.F. Bharucha, Bala Sundaram, and M.G. Raizen, *Phys. Rev. Lett.* **75**, 4598 (1995); C.F. Bharucha, J.C. Robinson, F.L. Moore, Bala Sundaram, Qian Niu, and M.G. Raizen, *Phys. Rev. E* **60**, 3881 (1999); B.G. Klappauf, W.H. Oskay, D.A. Steck, and M.G. Raizen, *Physica (Amsterdam)* **131D**, 78 (1999).
- [10] S. Fishman, D.R. Grempel, and R.E. Prange, *Phys. Rev. Lett.* **49**, 509 (1982); D.R. Grempel, R.E. Prange, and S. Fishman, *Phys. Rev. A* **29**, 1639 (1984).
- [11] F.M. Izrailev and D.L. Shepelyansky, *Sov. Phys. Dokl.* **24**, 996 (1979).
- [12] W.H. Oskay, D.A. Steck, V. Milner, B.G. Klappauf, and M.G. Raizen, *Opt. Commun.* **179**, 137 (2000).
- [13] M.V. Berry and E. Bodenschatz, *J. Mod. Opt.* **46**, 349 (1999).