Microscopic Correlation Functions for the QCD Dirac Operator with Chemical Potential

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A chiral random matrix model with complex eigenvalues is solved as an effective model for QCD with nonvanishing chemical potential. The new correlation functions derived from it are conjectured to predict the local fluctuations of complex Dirac operator eigenvalues at zero virtuality. The parameter measuring the non-Hermiticity of the random matrix is related to the chemical potential. In the phase with broken chiral symmetry all spectral correlations are calculated for finite matrix size N and in the large-N limit at weak and strong non-Hermiticity. The derivation uses the orthogonality of the Laguerre polynomials in the complex plane.

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Random matrix models (RMM) have been a useful tool in theoretical physics for a long time. In many physical systems the local fluctuation properties, for example of the energy levels, are universal and can be successfully described by RMM, where we refer to [1] for a review. Although in the generic situation the Hamiltonian and other physical observables are Hermitian thus having real eigenvalues, there exist also important cases where complex eigenvalues occur. As examples, we mention localization in superconductors [2], dissipation and scattering in quantum chaos [3], or quantum chromodynamics (QCD) with chemical potential [4].

Much less is known so far for spectral correlations of complex eigenvalues derived from RMM. Although the first results date back to Ginibre [5] where the correlations for the complex unitary ensemble labeled by the Dyson index $\beta = 2$ were calculated, progress has been slow. The correlation functions of the ensemble with real nonsymmetric matrices ($\beta = 1$) are still unknown. Results for quaternion matrices ($\beta = 4$) were obtained more recently in [6,7] and the inclusion of Dirac mass terms for $\beta = 2$ in [8]. Furthermore, it has been realized by works of Fyodorov and collaborators [9] that different regimes of complex eigenvalues exist, the weak and strong non-Hermiticity limit. In the present work we extend these results to a new chiral complex matrix model.

Chiral RMM of real eigenvalues have been introduced to describe the local fluctuation properties of the Dirac operator in QCD at the origin [10]. The low energy spectrum of the QCD Dirac operator is a very sensitive tool to study the phenomenon of chiral symmetry breaking [11]. The predictions of different chiral RMM ensembles have been very successful in describing the dependence on the gauge group and its representation, the number of quark flavors and masses, and topology [12]. In particular the topology dependence has been very useful in comparison with new developments in lattice gauge theory, admitting to incorporate an exact chiral symmetry [13]. By now the field theoretic origin of the RMM description has been well understood [14].

On the other hand lattice simulations in the presence of a chemical potential μ, which renders the Dirac eigenvalues complex, remain extremely difficult, as reviewed in [15]. Although recent progress has been made [16] along the phase transition line the general phase diagram remains unexplored for $\mu \neq 0$. In this context analytical knowledge of microscopic correlation functions from complex chiral RMM could be very useful, in view of its success in predicting real eigenvalue correlations.

Chiral RMM including the effect of μ have been already studied in several works [4,17–19]. These are schematic models for QCD with $\mu \neq 0$ sharing its global symmetries but lacking yet a derivation from field theory such as [14]. In [4] the nature of the quenched limit has been analyzed in a chiral RMM, and the global density of complex eigenvalues together with its boundary have been calculated as a function of μ. The phase diagram of QCD in the temperature density plane has been predicted from such a RMM [18]. Recently complex Dirac eigenvalues calculated on the lattice have been successfully confronted to a complex RMM on the microscopic scale given by the inverse volume in the bulk [19]. The nearest neighbor distribution along a given direction in the complex plane was considered and a transition from the Unitary to the Ginibre ensemble was observed at increasing μ, ending in a Poisson distribution. Our aim is to provide more detailed information to be compared with lattice data. Therefore we consider a new effective matrix model which does allow us to calculate all microscopic correlation functions in the complex plane, both in the limit of weak and strong non-Hermiticity. It is defined as a natural chiral extension of the complex ensemble treated in [8,9]. Although the main motivation is the application to QCD lattice calculations with $\mu \neq 0$ the new correlation functions we derive may find other applications as well, such as the fractional quantum Hall effect [20] or two-dimensional charged plasmas [21]. We restrict ourselves to work with a Gaussian chiral RMM. In particular, we will not touch the issue of universality here although we expect that in analogy to [22]

the same correlation functions hold for a more general weight function at large N.

We start our investigation by defining our model. The chiral random matrix partition function in terms of the complex eigenvalues $z_{j=1,\ldots,N}$ of a complex $N \times N$ matrix $J = H + i\sqrt{(1-\tau/1+\tau)}A$ is defined as

$$\mathcal{Z}_N^{(N_f,\nu)}(\tau) \equiv \int \prod_{j=1}^{N} dz_j dz_j^* w^{(a)}(z_j) |\Delta_N(z_1^2, \ldots, z_N^2)|^2, \quad (1)$$

where we have introduced the weight function

$$w^{(a)}(z) = |z|^{2a+1} \exp\left\{ -\frac{N}{1-\tau^2}\left[|z|^2 - \frac{\tau}{2}(z^2 + z^{*2}) \right] \right\}. \quad (2)$$

It captures the influence of N_f massless quark flavors in a fixed sector of topological charge ν, where $a = N_f + \nu$. We have taken the absolute value of the Dirac determinant but the weight depends on the real and imaginary parts of the eigenvalues. It has been chosen to be Gaussian with variance $(1+\tau)/2N$ for both the Hermitian and anti-Hermitian part of J, H, and A, respectively. The parameter $\tau \in [0, 1]$ controls the degree of non-Hermiticity and will be related to the chemical potential μ in QCD from [4] below. The Jacobi determinant from the diagonalization of J [5] yields the Vandermonde determinant $\Delta_N(z_1, \ldots, z_N) = \prod_{k>l}^{N}(z_k - z_l)$. The reason for the model Eq. (1) being chiral can be seen as follows. For real eigenvalues the chiral ensemble [10] is usually defined on the real positive line, $\int_0^\infty d\lambda \lambda^a \exp(-N\lambda)$. By substituting $\lambda = y^2$ it can be mapped to the full real axis, $\int_{-\infty}^{\infty} dy\, |y|^{2a+1} \exp(-Ny^2)$, where y corresponds to a real eigenvalue of the Dirac operator $\not{D}$ [23]. In the latter picture the continuation into the complex plane is straightforward, leading to the ensemble (1). The difference to the massless nonchiral ensemble [8] is the additional power $|z|$ and the argument of Δ_N. Furthermore, the orthogonal polynomials for the ensemble (1) are given by Laguerre polynomials $L_n^a(z^2)$ in the complex plane.

A chemical potential μ is included in the QCD action by adding $\mu\gamma_0$ to the Dirac operator $\not{D}$, making its eigenvalues complex. We aim to describe the local fluctuation of small eigenvalues close to the origin, because of their importance for chiral symmetry breaking through the Banks-Casher relation [24]. For μ not to completely dominate the Dirac determinant we will restrict ourselves to small values of μ. In [4] a chiral RMM different from Eq. (1) was used, replacing $\not{D}$ by a chiral random matrix and keeping the additional term $\mu\gamma_0$ explicitly. Here, we treat $\not{D} + \mu\gamma_0$ as a chiral, complex matrix instead. The advantage of our model is to have an eigenvalue representation and to permit for explicit calculations of all microscopic correlation functions. We will take the model [4] to relate our parameter τ to μ by comparing to the macroscopic spectral density $\rho(z)$ and its boundary calculated there as a function of μ. We conjecture that [4] and our model describe the same universal local fluctuations of complex QCD Dirac eigenvalues. In the limit of small μ the spectral density of [4] becomes approximately constant, $\rho(z) = 1/4\pi\mu^2$, and it is bounded by an ellipse, $x^2/4 + y^2/4\mu^2 = 1$, where $z = x + iy$. This behavior can also be observed for lattice data with small μ (e.g., in [19,25]). The macroscopic density in our model can be read off from [26] since the Dirac determinants are subdominant in the macroscopic large-N limit:

$$\rho(z) = \frac{1}{\pi(1-\tau^2)}, \quad \text{if } \frac{x^2}{(1+\tau)^2} + \frac{y^2}{(1-\tau)^2} \leq 1. \quad (3)$$

We therefore identify

$$4\mu^2 \equiv (1-\tau^2), \quad (4)$$

valid for small chemical potential and τ close to unity meaning small non-Hermiticity. For large values of μ the eigenvalue density on the lattice is no longer constant and develops a hole in the middle (see, e.g., [19,25]). We will see such a hole develop in the microscopic correlations at strong non-Hermiticity, as shown in Fig. 2.

After having identified all parameters in our model Eq. (1) we turn to its solution using the powerful method of orthogonal polynomials [6]. We only state the results and refer to [27] for details. All eigenvalue correlation functions are first given for a finite number of eigenvalues N and then in two different large-N limits corresponding to weak and strong non-Hermiticity. The orthogonal polynomials in the complex plane are defined as

$$\int dz dz^* w^{(a)}(z) P_k^{(a)}(z) P_l^{(a)}(z^*) = \delta_{kl}. \quad (5)$$

Following standard techniques [6] the knowledge of the kernel of orthogonal polynomials

$$K_N^{(a)}(z_1, z_2^*) \equiv [w^{(a)}(z_1) w^{(a)}(z_2^*)]^{1/2} \sum_{k=0}^{N-1} P_k^{(a)}(z_1) P_k^{(a)}(z_2^*) \quad (6)$$

allows one to calculate all k-point correlation functions

$$\rho_N^{(a)}(z_1, \ldots, z_k) = \det_{1 \leq i,j \leq k} [K_N^{(a)}(z_i, z_j^*)]. \quad (7)$$

The result for the orthogonal polynomials Eq. (5) is given in terms of Laguerre polynomials

$$P_k^{(a)}(z) \equiv \left[\binom{a+k}{k} f^{(a)}(\tau) \right]^{-1/2} (-\tau)^k L_k^a\left(\frac{Nz^2}{2\tau}\right), \quad (8)$$

with the normalization integral $f^{(a)}(\tau) \equiv \int dz dz^* w^{(a)}(z)$

$$f^{(a)}(\tau) = N^{-a-3/2}\pi;\Gamma\left(a + \frac{3}{2}\right)(1-\tau^2)^{(a/2)+(3/4)} \times P_{a+(3/2)}\left(\frac{1}{\sqrt{1-\tau^2}}\right) \quad (9)$$

and $P_{a+(3/2)}(x)$ being the Legendre function. All k-point correlation functions then follow by inserting the polynomials into Eqs. (6) and (7). In our results the parameter $a = N_f + \nu$ can be kept real (with $a > -1$). For example, we can set $a = -\frac{1}{2}$ as a check, recovering the even subset of the Hermite polynomials in the complex plane [20].

Since we did not find Eq. (8) in the literature we briefly sketch its derivation. Performing a change of variables $z \to e^{i\varphi} z$ in the normalization integral $f^{(a)}(\tau)$ we obtain

$$1 = \left\langle \exp\left[\frac{u}{u-1}\left(\frac{Nz^2}{2\tau}\right)\right]\exp\left[\frac{u^*}{u^*-1}\left(\frac{Nz^{*2}}{2\tau}\right)\right]\right\rangle, \quad (10)$$

with

$$u \equiv \frac{\tau^2(1-e^{2i\varphi})}{(1-\tau^2 e^{2i\varphi})} \quad (11)$$

and the average taken with respect to $w^{(a)}(z)$. Multiplying both sides of Eq. (10) with $[(1-u)(1-u^*)]^{-a-1}$ and recognizing the generating functional of the Laguerre polynomials we obtain the desired orthogonality relation, given properly normalized in Eq. (8).

After giving the exact solution for finite N we turn to the large-N limit. We first consider the weak non-Hermiticity limit. Following [9] we take the limit $\tau \to 1$ such that the combination

$$\lim_{N\to\infty} N(1-\tau^2) \equiv \alpha^2 = 4N\mu^2 \quad (12)$$

is kept fixed. Because of the identification Eq. (4) we consequently also rescale μ going to zero when $N \to \infty$. In other words the weak non-Hermiticity parameter α^2 directly measures the chemical potential in the microscopic scaling limit. Such a rescaling is similar to that of the quark masses [28]. It has been already mentioned in [17] that in a RMM the numerical effort to obtain convergence grows exponentially with $N\mu^2$. Keeping it fixed here should make a comparison to data feasible. Furthermore, we also rescale the complex eigenvalues keeping

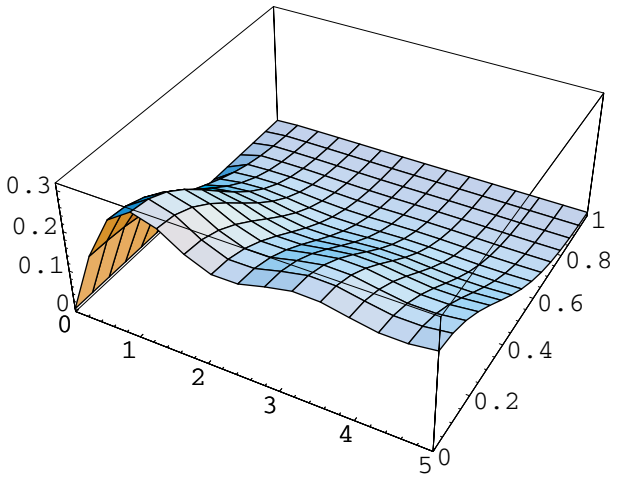

FIG. 1 (color online). The microscopic density for $\alpha^2 = 0.6$.

$$N(\Re e z + i\Im m z) = Nz \equiv \xi, \quad (13)$$

fixed. The matrix size N corresponds to the volume on the lattice. This defines our microscopic origin scaling limit in the complex plane. The kernel Eq. (6) and correlators Eq. (7) also have to be rescaled with the mean level spacing $1/N$ of the eigenvalues.

In order to obtain the microscopic kernel from Eq. (6) we replace the sum by an integral, $\sum_{k=0}^{N-1} \to N\int_0^1 dt$, where $t = \frac{k}{N}$, and use the asymptotic limit of the Laguerre polynomials to finally arrive at

$$K_S^{(a)}(\xi_1, \xi_2^*) = \frac{|\xi_1\xi_2^*|^{a+(1/2)}}{\sqrt{2\pi\alpha^2}(\xi_1\xi_2^*)^a} e^{-(1/\alpha^2)[(\Im m\xi_1)^2+(\Im m\xi_2^*)^2]} \times \int_0^1 dt e^{-\alpha^2 t} J_a(\sqrt{2t}\xi_1) J_a(\sqrt{2t}\xi_2^*). \quad (14)$$

The microscopic, weakly non-Hermitian correlation functions obtained from Eq. (7) are our first main result:

$$\rho_S^{(a)}(\xi_1, \ldots, \xi_k) = \prod_{l=1}^{k}\left[\frac{|\xi_l|}{\sqrt{2\pi\alpha^2}} e^{-(2/\alpha^2)(\Im m\xi_l)^2}\right] \det_{1\le i,j\le k}\left[\int_0^1 dt e^{-\alpha^2 t} J_a(\sqrt{2t}\xi_i) J_a(\sqrt{2t}\xi_j^*)\right]. \quad (15)$$

As an important check in the Hermitian limit $\alpha^2 \to 0$ corresponding to $\tau = 1$ the universal correlations [10,22] of the chiral RMM with real eigenvalues are reproduced. To give an example for Eq. (15) we have depicted the quenched microscopic density in part of the complex plane in Fig. 1. The other directions follow from symmetry. The oscillations known from the real case [10] spread into the complex plane, indicating the locations of the individual eigenvalues.

We now turn to the strong non-Hermiticity limit. In this limit $\tau \in [0, 1)$ and consequently also μ from Eq. (4) is kept fixed in the large-N limit. The eigenvalues are now rescaled with the square root of the volume [8,9],

$$\sqrt{N}\,(\Re e z + i\Im m z) = \sqrt{N}\, z \equiv \xi, \quad (16)$$

defining our microscopic origin limit at strong non-Hermiticity. In this limit the infinite sum in Eq. (6) can be evaluated using standard formulas for Laguerre polynomials, and we obtain for the kernel and the correlation functions in the strong limit (given up to a constant)

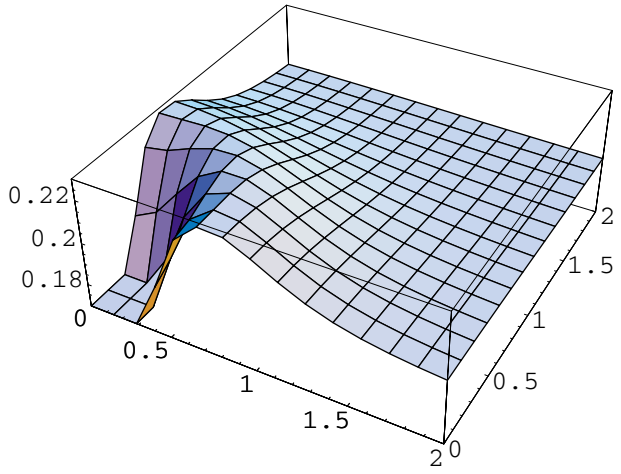

FIG. 2 (color online). The microscopic density for $\tau = 0.5$.

$$K_S^{(a)}(\xi_1, \xi_2^*) = \frac{2^a \Gamma(a+1)}{N^{a+(3/2)} f^{(a)}(\tau)(1-\tau^2)} \frac{|\xi_1 \xi_2^*|^{a+(1/2)}}{(\xi_1 \xi_2^*)^a} e^{\{-1/[2(1-\tau^2)]\}[|\xi_1|^2+|\xi_2^*|^2-(\tau/2)(\xi_1^{*2}-\xi_1^2+\xi_2^2-\xi_2^{*2})]} I_a\left(\frac{\xi_1 \xi_2^*}{1-\tau^2}\right), \tag{17}$$

$$\rho_S^{(a)}(\xi_1, \ldots, \xi_k) \sim \prod_{l=1}^{k} |\xi_l| e^{-1/(1-\tau^2)|\xi_l|^2} \det_{1 \le i,j \le k} \left[I_a\left(\frac{\xi_i \xi_j^*}{1-\tau^2}\right)\right]. \tag{18}$$

As a check they can be reobtained from Eq. (15) in the weak limit by taking $\alpha \to \infty$ there, identifying $\alpha^2 = 1 - \tau^2$. They differ from Ref. [21] due to the different interaction term in Eq. (1). An example for Eq. (18) is given in Fig. 2, the quenched microscopic density. As being observed in quenched lattice data [19,25] the microscopic spectral density develops a hole at the origin.

In summary, we have introduced a new chiral RMM having complex eigenvalues. All k-point correlation functions have been calculated explicitly at finite N as well as in the limits of weak and strong non-Hermiticity. The parameter τ that governs the non-Hermiticity has been related to the chemical potential μ in QCD by comparison to a schematic model with the same global symmetries. While the microscopic density shows qualitative features of Dirac eigenvalues calculated in lattice QCD a quantitative comparison with data remains to be done.

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