Quasi-Gaussian Statistics of Hydrodynamic Turbulence in $\frac{4}{3} + \epsilon$ Dimensions

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The statistics of two-dimensional turbulence exhibit a riddle: the scaling exponents in the regime of inverse energy cascade agree with the K41 theory of turbulence far from equilibrium, but the probability distribution functions are close to Gaussian-like in equilibrium. The skewness $S \equiv S_3(R)/S_2^{3/2}(R)$ was measured as $S_{\rm exp} \approx 0.03$. This contradiction is lifted by understanding that two-dimensional turbulence is not far from a situation with equipartition of enstrophy, which exists as true thermodynamic equilibrium with K41 exponents in space dimension of $d = \frac{4}{3}$. We evaluate the skewness $S(d)$ for $\frac{4}{3} \le d \le 2$, showing that $S(d) = 0$ at $d = \frac{4}{3}$, and that it remains as small as $S_{\rm exp}$ in two dimensions.

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Two-dimensional (2D) turbulence is not realized in nature or the laboratory, but only in computer simulations [1]. Nevertheless it serves as an idealized model for a variety of natural flow phenomena, such as geophysical flows in the atmosphere, oceans, and magnetosphere. Experimental setups that are close to 2D turbulence were realized in a number of laboratories [2], and the advent of faster computers allows precise simulations of the 2D Navier-Stokes equations [3]. In this Letter we are interested in the statistical characteristics of 2D turbulence, in the probability distribution functions, and in spectra of velocity differences. The velocity difference across a scale $\boldsymbol{R}$ is written in terms of the velocity field $\boldsymbol{u}(\boldsymbol{r}, t)$, $\boldsymbol{w}(\boldsymbol{r}, \boldsymbol{R}, t) \equiv \boldsymbol{u}(\boldsymbol{r} + \boldsymbol{R}, t) - \boldsymbol{u}(\boldsymbol{r}, t)$. Usually one measures the longitudinal component, $w_\ell(\boldsymbol{r}, \boldsymbol{R}, t) \equiv [\boldsymbol{u}(\boldsymbol{r} + \boldsymbol{R}, t) - \boldsymbol{u}(\boldsymbol{r}, t)] \cdot \boldsymbol{R}/R$, the probability distribution function (pdf) of this object, denoted as $P[w_\ell(\boldsymbol{r}, \boldsymbol{R})]$, as well as moments of this pdf, such as the second and third order structure functions

$$S_2(R) \equiv \langle w_\ell^2(\boldsymbol{r}, \boldsymbol{R}, t)\rangle, \qquad S_3(R) \equiv \langle w_\ell^3(\boldsymbol{r}, \boldsymbol{R}, t)\rangle. \tag{1}$$

Here the average is over space and time. In stationary homogeneous and isotropic ensembles the structure function depends on R only.

It is well known that in 2D turbulence the properties of these objects depend on whether R is larger or smaller than the scale L at which energy is injected into the system. For $R \gg L$ (but smaller than the outer boundaries of the system) one observes an inverse cascade of energy, and $S_2(R)$ is found to scale with an exponent in agreement with the Kolmogorov 1941 theory, i.e.,

$$S_2(R) = C_2(\varepsilon R)^{2/3}, \qquad R \gg L. \tag{2}$$

Here C_2 is a dimensionless coefficient of the order of unity, and ε is the mean energy flux per unit time and mass. For $R \ll L$ (but larger than the dissipative scale) one observes a direct cascade of enstrophy, with an exponent close to 2, but with some logarithmic corrections. Recent experiments and simulations lend strong support to Eq. (2); on the face of it this indicates that the system is very far from equilibrium, where equipartition of energy is expected. On the other hand, the same experiments and simulations indicate that $P[w_\ell(\boldsymbol{r}, \boldsymbol{R})]$ appears almost Gaussian, as if the system were very close to equilibrium. Quantitatively one measures the skewness

$$S = S_3(R)/S_2^{3/2}(R), \tag{3}$$

with the result $S_{\rm exp} \approx 0.03 \ll 1$. This seeming contradiction and its resolution are the subjects of this Letter.

The basic idea of this Letter is that there exists one space dimension for which these observations are not at all in contradiction. This is $d = \frac{4}{3}$, in which there exists an equilibrium state with equipartition of enstrophy, where the scaling expected for $S_2(R)$ is exactly (2), where all the odd moments and the cumulants of the even moments vanish, in particular, the skewness $S = 0$. By examining turbulence in $\frac{4}{3} + \epsilon$ dimensions, we establish that throughout the range $0 \le \epsilon \le \frac{2}{3}$ the situation remains very close to the one seen at $\epsilon = 0$; the value of $S(d)$ provides a natural small parameter to characterize the distance from $d = \frac{4}{3}$. This parameter remains very small up to two dimensions. Similarly, all the cumulants of the even moments and all the higher odd moments remain small. We thus interpret the statistics in the inverse cascade regime of 2D turbulence as a state very close to equilibrium.

It should be said at this point that attempts to connect turbulence to equilibrium statistical mechanics were made before. Well known are the theories advanced by Onsager [4], Hopf [5], and Lee [6], and see Ref. [1] for a review. Their ideas from equilibrium statistical mechanics were proposed to explain turbulence; as summarized in [1], the consensus is that such ideas were not relevant for forced turbulence but may be relevant for the final stages of the temporal evolution of decaying turbulence.

Next there were attempts to connect the physics in two dimensions to special properties of other, nonphysical dimensions. Of particular influence were ideas advanced by Fournier and Frisch [7], who identified $d \approx 2.05$ as the dimension at which the direction of the energy flux changes sign. It was hoped that this may provide a convenient starting point for perturbative expansions in the magnitude of the flux. A similar point was identified in shell models of turbulence, as a function of a parameter [8]. It turned out, however, that the fluctuations were not small at that point, the flux changed sign discontinuously, and perturbative theories did not yield useful insights. In our own work on shell models [9] we identified another point in parameter space where a quasiequilibrium state with equipartition of "enstrophy" coincided with an energy spectrum in agreement with Kolmogorov scaling exponents as in (2). This has led us to seek a one-parameter lifting of the Navier-Stokes equations where a similar phenomenon exists. We were thus led to examine turbulence in $d = \frac{4}{3}$ and its vicinity. We argue below that there is a fundamental difference between the properties of turbulence in $d = \frac{4}{3}$ (where the flux vanishes) and $d \approx 2.05$ (where the flux discontinuously changes sign). At the former point the statistics is exactly Gaussian, with equipartition of the enstrophy. The latter has no such distinction, and the statistics may differ strongly from Gaussian.

We begin by generalizing 2D turbulence to $d < 2$ dimensions. In doing so we preserve the d-dimensional energy and enstrophy, in distinction with other such extensions that fail to do so. The Navier-Stokes equations for the incompressible velocity field $\boldsymbol{u}(\boldsymbol{r}, t)$ read

$$\partial \boldsymbol{u}/\partial t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu \nabla^2 \boldsymbol{u} + \nabla p = \boldsymbol{f}, \tag{4}$$

where ν is the kinematic viscosity, p is the pressure, and $\mathbf{f}$ is a force which maintains the flow. In $d = 2$ it is natural to consider the vorticity field $\boldsymbol{\omega}(\boldsymbol{r}, t) = \nabla \times \boldsymbol{u}(\boldsymbol{r}, t)$. For $\nu = \boldsymbol{f} = 0$ the curl of Eq. (4) is $\partial \boldsymbol{\omega}/\partial t = \nabla \times (\boldsymbol{u} \times \boldsymbol{\omega})$. We introduce the right-handed coordinate system ($x_1 = x$, $x_2 = y$, and $x_3 = z$) in which for 2D flows $u_3 = 0$ and $\boldsymbol{u}(\boldsymbol{r}, t) = \boldsymbol{u}(\boldsymbol{x}, t)$, where $\boldsymbol{x} = \{x_1, x_2\}$. The vorticity has the single component ω_3 for which Eq. (4) reduces to

$$\partial \omega/\partial t + (\boldsymbol{u} \cdot \nabla)\omega = 0. \tag{5}$$

This equation has two quadratic integrals of motion, the (kinetic) energy E and the enstrophy H:

$$E = \frac{1}{2}\int u^2(\boldsymbol{x}, t) d^2x, \qquad H = \frac{1}{2}\int \omega^2(\boldsymbol{x}, t) d^2x. \tag{6}$$

The velocity and vorticity of a 2D flow may be derived from the stream function $\psi(\boldsymbol{x}, t)$: $\boldsymbol{u}(\boldsymbol{x}, t) = -\nabla \times \hat{\boldsymbol{z}}\psi(\boldsymbol{x}, t)$, $\omega(\boldsymbol{x}\boldsymbol{x}, t) = -\nabla^2\psi(\boldsymbol{x}, t)$, where $\hat{\boldsymbol{z}}$ is a unit vector orthogonal to the $\boldsymbol{x}$ plane, and ∇^2 is the Laplacian operator in the plane. In $\boldsymbol{k}$ representation, $a(\boldsymbol{k}, t) \equiv k \int d\boldsymbol{r} \exp[-i(\boldsymbol{r} \cdot \boldsymbol{k})]\psi(\boldsymbol{r}, t)$. The Fourier transforms of $\boldsymbol{u}(\boldsymbol{x}, t)$ and of $\boldsymbol{\omega}(\boldsymbol{x}, t)$, respectively, $\boldsymbol{u}(\boldsymbol{k}, t)$ and $\boldsymbol{\omega}(\boldsymbol{k}, t)$, are expressed in terms $a(\boldsymbol{k}, t)$: $\boldsymbol{u}(\boldsymbol{k}, t) = i(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{k}})a(\boldsymbol{k}, t)$, $\omega(\boldsymbol{k}, t) = -ka(\boldsymbol{k}, t)$, where $\hat{\boldsymbol{k}} = \boldsymbol{k}/k$. Now, by Eq. (5)

$$\begin{aligned}
&\frac{\partial a(\boldsymbol{k}, t)}{\partial t} = \frac{1}{2}\int \frac{d^2q d^2p}{(2\pi)^2}\delta(\boldsymbol{k} + \boldsymbol{q} + \boldsymbol{p})V_{kqp}a^*(\boldsymbol{q}, t)a^*(\boldsymbol{p}, t),\\
&V_{kqp} = S_{kqp}(p^2 - q^2)/2kqp, \qquad S_{kqp} \equiv 2qp\sin\phi_{pq},\\
&S_{kqp} = S_{pkq} = S_{qpk} = -S_{kpq} = -S_{qkp} = -S_{pqk},\\
&|S_{kpq}| = \sqrt{2(k^2q^2 + q^2p^2 + p^2k^2) - k^4 - q^4 - p^4}.
\end{aligned} \tag{7}$$

Here the interaction amplitude (or "vertex") V_{kqp} is expressed via S_{kqp}; $|S_{kqp}|/4$ is the area of the triangle formed by the vectors $\boldsymbol{k}$, $\boldsymbol{q}$, and $\boldsymbol{p}$. $\phi_{pq} = \phi_p - \phi_q$; ϕ_k, ϕ_q, and ϕ_p are the angles in the triangle plane between the x_1 axis and the vectors $\boldsymbol{k}$, $\boldsymbol{q}$, and $\boldsymbol{p}$, respectively.

The vertex V_{kqp} satisfies two Jacoby identities [1]

$$(V_{kqp} + V_{pkq} + V_{qpk}) = 0, \tag{8}$$

$$(k^2V_{kqp} + p^2V_{pkq} + q^2V_{qpk}) = 0. \tag{9}$$

The first one guarantees the conservation of energy in the inviscid forceless limit, while the second conserves the enstrophy. In terms of $a(\boldsymbol{k}, t)$ Eqs. (6) read

$$E = \int \frac{|a(\boldsymbol{k}, t)|^2 d^2k}{2(2\pi)^2}, \qquad H = \int \frac{k^2|a(\boldsymbol{k}, t)|^2 d^2k}{2(2\pi)^2}. \tag{10}$$

To generalize to d dimensions we keep the vertices unchanged but integrate over d^dqd^dp in Eq. (7). The 2nd and 3rd order correlation functions of a, a^* are

$$(2\pi)^d\delta(\boldsymbol{k} + \boldsymbol{q})n_k(t) = \langle a(\boldsymbol{k}, t)a(\boldsymbol{q}, t)\rangle, \tag{11}$$

$$(2\pi)^d\delta(\boldsymbol{k} + \boldsymbol{q} + \boldsymbol{p})F_{kqp}(t) = \langle a(\boldsymbol{k}, t)a(\boldsymbol{q}, t)a(\boldsymbol{p}, t)\rangle. \tag{12}$$

In isotropic systems, we do not need to carry the boldface $\boldsymbol{k}$ index in n_k and in F_{kqp}. In terms of n_k we define the volume densities in d dimensions:

$$\mathcal{E} \equiv \frac{E}{V} = \int \frac{d^dk}{2(2\pi)^d} n_k, \qquad \mathcal{H} \equiv \frac{H}{V} = \int \frac{d^dk}{2(2\pi)^d} k^2 n_k.$$

Since n_k and k^2n_k serve as the energy and enstrophy densities, respectively, in $\boldsymbol{k}$ space, the thermodynamic equilibrium can be achieved with equipartition in any of these quantities:

$$\begin{aligned}
&n_k^{\varepsilon 0} = A_\varepsilon, && \text{energy equipartition;}\\
&n_k^{h0} = A_h/k^2, && \text{enstrophy equipartition.}
\end{aligned} \tag{13}$$

In such a state of thermodynamic equilibrium unavoidably all fluxes vanish and the pdf of the velocity differences is Gaussian. Usually in the theory of turbulence one rather considers flux equilibria, which in the present situation can be with energy flux or enstrophy flux. Dimensional analysis for such flux equilibria predicts

$$n_k^\varepsilon = C_\varepsilon(d)\varepsilon^{2/3}k^{-x_\varepsilon}, \quad x_\varepsilon = d + \frac{2}{3}, \quad \text{energy flux;} \tag{14}$$

$$n_k^h = C_h(d)h^{2/3}k^{-x_h}, \quad x_h = d + 2, \quad \text{enstrophy flux.} \tag{15}$$

Here h is the mean enstrophy flux per unit time and mass, whereas $C_\varepsilon(d)$ and $C_h(d)$ are d-dependent dimensionless coefficients. In terms of $S_2(R)$ these results are in agreement with (2) for all d for energy flux equilibrium. For enstrophy flux equilibrium the result is $S_2(R) \propto R^2$. The basis for further development is the immediate observation that for $d = \frac{4}{3}$ the scaling exponent for energy flux equilibrium, $x_\varepsilon = 2$, coincides with the exponent 2 of the equipartition of enstrophy. Accordingly for $d = \frac{4}{3}$ the law (2) is in agreement with enstrophy equipartition and therefore also with a Gaussian pdf for the velocity differences. In the rest of this Letter we show that in dimensions $\frac{4}{3} < d \le 2$, the flux remains small (for given total energy of the system) and the pdf's do not change much from Gaussianity. For the sake of brevity we consider the skewness as a measure of the deviation from Gaussianity; similar results can be derived for any odd moments or any cumulant of even moment.

The skewness (3) is now d dependent, $S(d)$. To compute it we need to separately find $S_2(R)$ and $S_3(R)$. We start with the former. The structure function $S_2(R)$ can be computed from Eq. (14). In two dimensions

$$S_2(R) = \int \frac{d^2k}{(2\pi)^2} |\exp(ikR\cos\phi_k) - 1|^2 \sin^2\phi_k n_k^\varepsilon = C_\varepsilon(\varepsilon R)^{2/3} A_2, \tag{16}$$

$$A_2 = \int_0^{2\pi} \frac{\sin^2\phi_k d\phi}{2\pi^2} \int_0^\infty \frac{d\kappa}{\kappa^{5/3}} [1 - \cos(\kappa\cos\phi_k)] \approx 0.0855. \tag{17}$$

In d dimensions we write $S_2(R) = C_2(d)(\varepsilon R)^{2/3} = C_\varepsilon(d)(\varepsilon R)^{2/3} A_2(d)$. Analyzing the d-dimensional generalization of (17) one proves that $A_2(d)$ is not critical at $d = \frac{4}{3}$. On the other hand, we will show that $C_\varepsilon(d)$ [and therefore $S_2(R)$] diverges when $d \to \frac{4}{3}$. Therefore we will estimate $A_2(d)$ by its value A_2 at two dimensions, $C_2(d) = C_\varepsilon(d)A_2$. This introduces at most an error of the order of unity. On the one hand, $S_3(R) = 12\varepsilon R/d(d+2)$ exactly. To compute $C_\varepsilon(d)$ we will use this result and the relation of $S_3(R)$ to F_{kqp}:

$$\begin{aligned} S_3(R) = &\int \frac{d^dk d^dq d^dp}{(2\pi)^{2d}} \delta(\boldsymbol{k} + \boldsymbol{q} + \boldsymbol{p}) \sin\phi_k \sin\phi_q \sin\phi_p \\ &\times \mathrm{Im}\{[\exp(ikR\cos\phi_k) - 1][\exp(iqR\cos\phi_q) - 1][\exp(ipR\cos\phi_p) - 1]\} F_{kqp}. \end{aligned} \tag{18}$$

This is as far as we can proceed exactly. Now we will express the third order correlator F_{kqp} in terms of the second order n_k. It is well known that this cannot be done without closure approximations. The latter are known to provide semiquantitative estimates of the coefficients of correlation functions, and in the present context we expect such approximations to perform better than in 3D due to the existence of the small parameter $S(d)$ that we will expose in this calculation. The starting point is the equation of motion of $n_k(t)$, which can be exactly written in terms of the third order correlation F_{kqp}:

$$\begin{aligned} &\partial n_k / 2\partial t = I_k, \\ &I_k = (2\pi)^{-d} \int d^dq d^dp \delta(\boldsymbol{k} + \boldsymbol{q} + \boldsymbol{p}) V_{kqp} F_{kqp}. \end{aligned} \tag{19}$$

A standard closure approximation expresses F_{kqp} via n_k. Proper closures in turbulence involve the following steps: first, one considers the sweeping-free renormalized perturbation theory to first order. Second, one assumes a simple analytic form for the time decay of correlation and response functions. A typical result reads [10,11]

$$\begin{aligned} F_{kqp} &= N_{kqp}\theta_{kqp}, \\ N_{kqp} &= V_{kqp}n_q n_p + V_{pkq}n_k n_q + V_{qpk}n_p n_k. \end{aligned} \tag{20}$$

Here θ_{kqp} is a closure dependent "triad-decorrelation time"; assuming simple exponential decay for the second order correlation function and response function, one finds

$$\theta_{kqp} = 1/(\gamma_k + \gamma_q + \gamma_p). \tag{21}$$

In Eq. (21) γ_k is the width of the assumed Lorentzian line shape. The latter can be estimated as follows: $\gamma_k = A_\gamma(d)k\sqrt{S_2(1/k)} = C_\gamma(d)\varepsilon^{1/3}k^{2/3}$. Here $A_\gamma(d)$ is a coefficient of the order of unity. Since $S_2(R)$ diverges at $d = \frac{4}{3}$ we will evaluate below $A_\gamma(d)$ via its two-dimensional value A_γ:

$$C_\gamma(d) = A_\gamma\sqrt{C_2(d)}. \tag{22}$$

We stress that although Eq. (20) is derived using usual uncontrolled closure approximations, once it is substituted into Eq. (19) the latter conserves the energy and enstrophy invariants $\mathcal{E}$ and $\mathcal{H}$ defined in all d dimensions. This distinguishes our analysis from some previous theories like [7] which conserved enstrophy in two dimensions only. Note that our equations of motion exhibit the equilibria (13)–(15) as exact results. Thermodynamic equilibria follow directly from the Jacoby identities (8) and (9) which yield $N_{kqp} = F_{kqp} = 0$ and hence also $I_k = 0$. To show that also the flux equilibria are satisfied exactly one needs to use the Kraichnan-Zakharov transformation [12] and the Jacoby identities. There F_{kqp} does not vanish in general.

We now note that in dimension $d = \frac{4}{3}$ when $n_k \propto 1/k^2$, F_{kqp} vanishes by itself. This follows from the fact that N_{kqp} vanishes due to the Jacoby identity (9). It is worthwhile to stress that in our context this result is derived only to first order in renormalized perturbation theory. It is, however, a stronger result which can be established order by order to all orders.

Substituting (20) with $n_k = n_k^\varepsilon$ of Eq. (14), and Eq. (21), we can rewrite $S_3(R)$ in the form

$$S_3(R) = [C_\varepsilon(d)]^2 A_3(d)(\varepsilon R)/C_\gamma(d), \tag{23}$$

$$A_3(d) = \int \frac{d^d\tilde{k}d^d\tilde{q}d^d\tilde{p}}{(2\pi)^{2d}} \frac{\delta(\tilde{\boldsymbol{k}} + \tilde{\boldsymbol{q}} + \tilde{\boldsymbol{p}})S_{\tilde{k}\tilde{q}\tilde{p}}}{\tilde{k}^{2/3} + \tilde{q}^{2/3} + \tilde{p}^{2/3}} \frac{\sin\phi_k \sin\phi_q \sin\phi_p}{(\tilde{k}\tilde{q}\tilde{p})^{2d+5/3}}$$
$$\times [\sin(\tilde{k}\cos\phi_k) + \sin(\tilde{q}\cos\phi_q) + \sin(\tilde{p}\cos\phi_p)]2[\tilde{k}^{d+2/3}(\tilde{p}^2 - \tilde{q}^2) + \tilde{p}^{d+2/3}(\tilde{q}^2 - \tilde{k}^2) + \tilde{q}^{d+2/3}(\tilde{k}^2 - \tilde{p}^2)]. \tag{24}$$

Combining Eqs. (3), (16), (22), and (23) we find $S(d) = A_3(d)/A_2^2 A_\gamma$. Obviously, $A_3(d) = 0$ at $d = \frac{4}{3}$. This implies that $S_2(R)$ [i.e., $C_\varepsilon(d)$] diverge at this point. It is easy to prove that the first derivative of $A_3(d)$ with respect to d at $d = \frac{4}{3}$ is finite. We can therefore approximate $A_3(d)$ as

$$A_3(d) \approx a(d - 4/3), \tag{25}$$

up to orders of $(d - \frac{4}{3})^2$. On the other hand, by direct numerical integration we found

$$A_3 \equiv A_3(d)|_{d=2} \approx 3.556 \times 10^{-4}. \tag{26}$$

Thus in two dimensions we estimate

$$S \equiv S(d)|_{d=2} \approx 0.0486/A_\gamma. \tag{27}$$

The experimental observation is $S_{\rm exp} \approx 0.03$. Taking liberty to use this finding we estimate $A_\gamma \approx 1.62$. This is in agreement with our expectation that A_γ is of $O(1)$. Now we can estimate $S(d)$ in the whole range $\frac{4}{3} \le d \le 2$ by using the linear approximation (25) and find

$$S(d) \approx \frac{0.0729}{A_\gamma}\left(d - \frac{4}{3}\right) \approx 0.045\left(d - \frac{4}{3}\right). \tag{28}$$

The main conclusion of this Letter is that although $d = 2$ is finitely removed from $d = \frac{4}{3}$, the relevant small parameter remains small all the way to $d = 2$, because of the numerical smallness of the ratio $A_3/A_2^2 \approx 0.0486$. This smallness stems from generic geometric cancellations in the last line of the integrand in (24). This originates from the structure of the vertex V_{kqp} and is therefore fundamental to Euler equations in two dimensions. We stress at this point that other measures for the deviations from Gaussianity display similar smallness, as will be shown in a forthcoming publication.

In summary, we have addressed the experimental findings of the statistics of 2D turbulence. It is hardly surprising that 2D turbulence is not strongly intermittent; in the inverse cascade regime there are no mechanisms to create rare events that are intimately related to the sharpening and enhancement of fluctuations as they are transferred *down* the scales in generic 3D direct cascades [13]. On the other hand, the fact that the statistics of 2D turbulence are so close to Gaussian came as a major surprise. In this Letter we offered an explanation to this finding. We have identified $d = \frac{4}{3}$ as a convenient point around which to develop a theory of 2D turbulence. The statistics there are Gaussian, but the spectrum in the inverse energy flux regime is K41. The skewness is zero, allowing a sensible closure theory for d slightly larger than $\frac{4}{3}$. Estimating the skewness as a function of d, we are led to conclude that it remains small also in two dimensions. We can thus interpret 2D turbulence as a state close to equilibrium. In future work we will examine further the structure of the theory in the range $\frac{4}{3} \le d \le 2$ to assess further the quality of the closure approximation.

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[1] R. H. Kraichnan and D. Montgomery, Rep. Prog. Phys. **43**, 547 (1980).
[2] See P. Tabeling, Phys. Rep. **362**, 1 (2002), and references therein.
[3] G. Boffetta, A. Celani, and M. Vergassola, Phys. Rev. E **61**, R29 (2000).
[4] L. Onsager, Nuovo Cimento Suppl. **6**, 279 (1949).
[5] E. Hopf, J. Tat. Mech. Anal. **1**, 87 (1952).
[6] T. D. Lee, Q. Appl. Math. **10**, 69 (1952).
[7] J.-D. Fournier and U. Frisch, Phys. Rev. A **17**, 747 (1978).
[8] P. Giuliani, M. H. Jensen, and V. Yakhot, Phys. Rev. E (to be published).
[9] T. Gilbert, V. S. L'vov, A. Pomyalov, and I. Procaccia, "Inverse Cascade Regime in Shell Models of 2-Dimensional Turbulence," nlin.CD/0201020 (to be published).
[10] R. H. Kraichnan, J. Fluid Mech. **47**, 525 (1971), and references therein.
[11] V. S. L'vov, Yu. L'vov, A. C. Newell, and V. E. Zakharov, Phys. Rev. E **56**, 390 (1997). There are two differences between Eq. (4.43) in this reference and Eq. (19) in the present Letter: $\omega_0(k)$ is identically zero here, and the vertex V has different symmetry properties and therefore it appears in slightly different combinations.
[12] V. E. Zakharov, V. S. L'vov, and G. Falkovich, *Kolmogorov Spectra of Turbulence* (Springer, Berlin, 1992).
[13] V. S. L'vov, A. Pomyalov, and I. Procaccia, Phys. Rev. E **63**, 056118 (2001).