Turbulentlike Fluctuations in Quasistatic Flow of Granular Media

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We analyze particle velocity fluctuations in a simulated granular system subjected to homogeneous quasistatic shearing. We show that these fluctuations share the following scaling characteristics of fluid turbulence in spite of their different physical origins: (i) scale-dependent probability distribution with non-Gaussian broadening at small time scales; (ii) spatial power spectrum of the velocity field showing a power-law decay, reflecting long-range correlations and the self-affine nature of the fluctuations; and (iii) superdiffusion of particles with respect to the mean background flow.

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The key role of fluctuations in quasistatic (QS) flow of granular media has been noted by several authors referring basically to stress fluctuations in time or the inhomogeneous distribution of forces in space [1,2]. Amazingly, few studies have been reported about the fluctuations of particle velocities under macroscopically homogeneous strain conditions. Velocity fluctuations have been observed to occur in a correlated fashion, though their scaling properties have not yet been analyzed [3,4]. Other recent studies concern mainly Couette flows where the strain is localized in the vicinity of the inner rotating cylinder [5].

In this Letter, we present a numerical investigation of velocity fields in a slow granular flow where the homogeneity of shearing is ensured by means of biperiodic boundary conditions. Thereby we obtain reliable statistics that allows for an accurate analysis of fluctuations. The occurrence of these fluctuations recalls the field of fluid turbulence. There obviously the physics is fundamentally different from that governing granular media. Nevertheless, the rich body of work devoted to the statistical analysis of the fluctuating part of the velocity field in fluid turbulence (cf. e.g., [6]) provides a suitable framework that can be applied in order to characterize the analogous fluctuating part of the velocity field in a granular medium. This point leads quite naturally to an interesting question: Are granular fluctuations *turbulent* in terms of scaling features such as the non-Gaussian broadening of probability density functions (pdf's), power-law spectrum in space, and anomalous diffusion? Although deeply rooted in fluid dynamics (Navier-Stokes equations and inertia regime), these key aspects of fluid turbulence *may*, in principle, prove to be relevant as well within a different physical context such as granular flows.

We will show that particle velocity fluctuations in our QS granular flows exhibit indeed strikingly similar features. A strict analogy with fluid turbulence makes certainly not much sense because of a drastically different physics that underlies these fluctuations. In particular, inertia effects are basically irrelevant in a QS granular flow, and frictional and hard-core inelastic interactions between particles have little in common with molecular interactions in a fluid. But, precisely because of these dissimilarities, the observed analogy in terms of scaling properties is quite nontrivial, and it might lead to new insights in both fields. In the following, we first describe the simulated granular system and our procedures of data analysis. Then, we present our main results focusing on the pdf's, correlations, and diffusion, respectively.

The investigated granular model is a two-dimensional assembly of 4000 frictional disks with diameters uniformly distributed between $D_{\min}$ and $D_{\max}$ with $D_{\max} = 3D_{\min}$. The particles interact through a stiff linear repulsive force as a function of mutual overlaps and the Coulomb friction law. The coefficient of friction is 0.5. The equations of motion for particle displacements and rotations are integrated by means of a predictor-corrector scheme [7].

An accurate evaluation of the statistics of fluctuations requires long-time homogeneous and steady shearing. However, ordinary wall-type boundary conditions induce a pronounced layering effect and the corners enhance the local frustrations whereby large strain and stress inhomogeneities arise when the box shape changes. System-size inhomogeneities may also occur due to shear localization. In order to circumvent such unwanted effects, we used biperiodic boundary conditions following a method similar in spirit to that devised by Parrinello and Rahman [8].

In our simulations, the gravity was set to zero and a confining pressure was applied along the y direction. The width L of the simulation cell was kept constant. The displacement field is decomposed into an affine displacement field $\delta \boldsymbol{r}^i \equiv (\delta r_x^i, \delta r_y^i)$ and a fluctuating field $\delta \boldsymbol{s}^i \equiv (\delta s_x^i, \delta s_y^i)$ of zero mean ($\langle \delta \boldsymbol{s} \rangle = 0$) carried by the particles i. The system is driven by imposing $\delta r_x^i = \delta t \gamma r_y^i$, where γ is a constant shear rate and δt is the time step. In other words, the Fouier mode $k = 0$ of the total strain is imposed, corresponding to a large-scale forcing. This driving mode was applied on a dense packing prepared by isotropic compaction. The height of the packing increases (dilation) in the initial stages of shearing before a homogeneous steady state is reached where volume changes fluctuate around zero. The focus of this Letter is the field $\delta \boldsymbol{s}^i$ which corresponds to a spatially periodic motion of the particles with respect to the background flow $\delta \boldsymbol{r}^i$.

Although our dynamic simulations involve the physical time, the inertial effects are negligibly small and the granular texture evolves quasistatically at time scales well below γ^{-1}. We normalize all times by γ^{-1} so that the dimensionless time t in what follows will actually represent the cumulative shear strain. We will also use the mean particle diameter D to scale displacements. As a result, the velocities will be scaled by γD and the power spectra in space by $(D^2\gamma)^2$. In our simulations the time step is $\delta t \simeq 10^{-7}$, and more than 2×10^7 steps are simulated, corresponding to a total strain larger than 2. The solid fraction in the steady state fluctuates in the range $[0.79, 0.81]$ and the average coordination number is 3.8.

It is important to emphasize here that the granular nature of our system does not allow us to apply exactly the same procedures of data analysis as in fluid turbulence. Turbulence studies focus mainly on velocity differences δv measured at a fixed point of a fluid over a time interval τ or between two points separated by a distance r. In contrast, the particle-scale granular motion involves a *discrete* displacement field that is carried by individual *particles*. Thus, our natural framework is Lagrangian rather than Eulerian. These differences are certainly important for a one-to-one comparison, but here we basically consider turbulence as a reference field from which we extract tools to characterize granular fluctuations.

Another distinctive feature of granular flow is that, due to collisions, the velocities are discontinuous in time. As the positions are better behaved, we characterize the fluctuating motions of the particles by "tracer" velocities defined from particle displacements δs^i by

$$\boldsymbol{v}^i(t, t+\tau) = \frac{1}{\tau}\int_t^{t+\tau} \delta \boldsymbol{s}^i(t')dt', \qquad (1)$$

where τ is the time resolution. Since we are concerned with steady flow, the statistical properties of $\boldsymbol{v}$ are independent of t (though, as shown below, they crucially depend on τ). Hence, accurate statistics can be obtained by cumulating the data from different time slices of a single simulation running for a long time.

Figure 1 shows a snapshot of fluctuating velocities $\boldsymbol{v}^i$ for a short time lag $\tau = 10^{-7}$. We see that large-scale well-organized displacements coexist with a strongly inhomogeneous distribution of amplitudes and directions on different scales. Eddylike structures (though without the singular vorticity concentrated in the core of these eddies) appear quite frequently, but they survive typically for strains τ less than 10^{-3}. After such short times, large-scale eddies break down and new (statistically uncorrelated) structures appear. This behavior is radically different from turbulence where the eddies survive long enough to undergo a significant distortion due to fluid motion [6].

FIG. 1. A snapshot of particle displacements $\delta \boldsymbol{s}^i$ with respect to the mean background flow.

We consider now the pdf's of the components $v_x^i(t, t+\tau)$ and $v_y^i(t, t+\tau)$ as a function of time resolution τ. The pdf's of v_y^i are shown in Fig. 2 for a short integration time $\tau = 10^{-3}$, and for a long integration time $\tau = 10^{-1}$. The pdf has changed from a nearly Gaussian shape at large τ to a non-Gaussian shape with broad stretched exponential tails extending nearly to the center of the distribution at small τ.

In order to characterize this non-Gaussian broadening of the pdf's, we calculated the flatness $F = \langle v_y^4\rangle/\langle v_y^2\rangle - 3$,

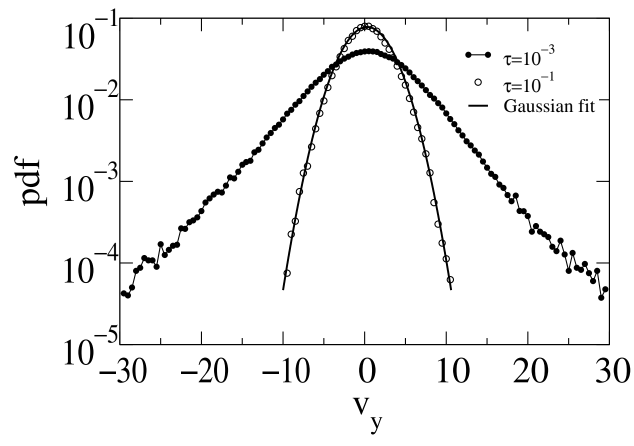

FIG. 2. The pdf's of the y components of fluctuating velocities for two different integration times: 10^{-3} (broad curve) and 10^{-1} (narrow curve). The latter is fitted by a Gaussian. The error bars are too small to be shown.

which is zero for a Gaussian distribution and 3 for a purely exponential distribution. The values of F as a function of τ, shown in Fig. 3, are consistent with zero at large τ ($\tau > 10^{-1}$) and rise to 5 for our finest time resolution ($\tau = 10^{-7}$) . A strictly similar behavior was observed for the component v_x.

The broadening of the exponential tails of the pdf's at increasingly smaller scales is a hallmark of fully developed turbulence (for velocity differences) [9]. It is attributed to the phenomenon of intermittency, i.e., strong localized energy transfers at small scales. In a QS granular flow, the basic physical mechanism underlying the fluctuations is the mismatch of the uniform strain field applied at the boundaries or in the bulk, with mutual exclusions of the particles. As a result, the local strains deviate from the mean (global) strain. The observation of a transition toward a Gaussian distribution for large time lags is a sign of loss of correlation and/or exhaustion of large fluctuations in the increment of displacement which occur at different times. Unfortunately, the rich multifractal scaling of velocity fluctuations is out of reach within the present investigation due to demanding statistics [10]. The analogy with turbulence, however, suggests further study along such routes.

In order to quantify the extent of correlations in space, we estimated the power spectrum E of velocity fluctuations both along and perpendicular to the flow and at different times. The Fourier transform was performed over the fluctuating velocity field defined on a fine grid by interpolating the velocities from particle centers. The power spectra were quite similar along and perpendicular to the flow, and for different snapshots of the flow. The averaged spectrum on one-dimensional cross sections is shown in Fig. 4. It has a clear power-law shape $k^{-\beta}$ ranging from the smallest wave number $k = D/L$, corresponding to scale L, up to a cutoff around $k = 0.5$, corresponding to nearly two particle diameters. The exponent is $\beta \simeq 1.24 \simeq 5/4$ over 1 decade (to be compared with the exponent $5/3$ as a hallmark of 3D turbulence for the spectrum of velocity differences). This means in practice that the fluctuating velocity field is self-affine with a Hurst exponent $H = (\beta - 1)/2 = 0.12$ [11].

Because of the peculiar behavior of the velocity field (discontinuous in time), one might expect that the power spectrum is sensitive to the time resolution τ. However, we checked that the value of β is independent of τ. It is also noteworthy that the presence of long-range correlations in displacements, reflected in the value of β, is in strong contrast with the observed correlation lengths of nearly $10D$ for contact forces [2].

The long-time behavior may be studied by considering the effective diffusion of the particles. Normal diffusion implies that the root-mean-square (rms) relative displacements λ in a given direction varies in proportion to the square root of time. In 3D fluid turbulence at high Reynolds numbers, the long-time pair diffusivity of suspended particles is anomalous, following the Richardson law $\lambda \propto \tau^{3/2}$, reflects the Kolmogorov-Obukhov velocity spectrum [6].

We analyzed the *kinematic* diffusion of single particles in our QS granular flow. One example of a single particle trajectory with respect to the background strain is shown in Fig. 5. We see that the fluctuating displacement is of the order of the mean particle diameter for a strain of the order of unity. Figure 6 shows the rms relative displacement $\lambda(\tau)$ of all particle pairs initially in contact, as a function of time along x and y directions. This clearly corresponds to a superdiffusion behavior$\lambda \propto \tau_{\alpha}$, with $\alpha \simeq 0.9$ for both components over nearly 3 decades of strain. Particle self-diffusivities exhibit a similar law. Since large-scale structures are short lived, this anomalous diffusion scaling cannot be solely attributed to velocity correlations. It reveals, above all, the long-time configurational memory of a granular medium in QS flow.

We note that Fig. 6 shows no anisotropy for the diffusion. In fact, the steric exclusion effects dominate over the large-scale strain field for small diffusive displacements (below one particle diameter), and hence the anisotropy may be weak at such scales. Moreover, we observe no crossover to normal scaling within the investigated strain range. We cannot exclude that for larger strains, when two

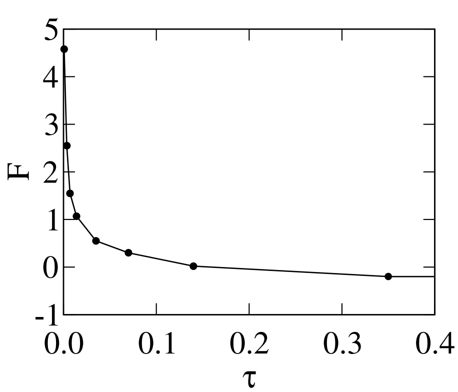

FIG. 3. Flatness F of the distribution of velocity fluctuations as a function of the integration time τ.

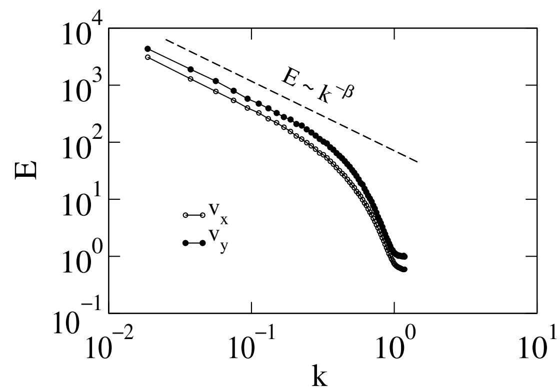

FIG. 4. Averaged power spectrum of the x and y components of the fluctuating velocity field with $\tau = 10^{-7}$ for one-dimensional cross sections along the mean flow. For the units, see the text.

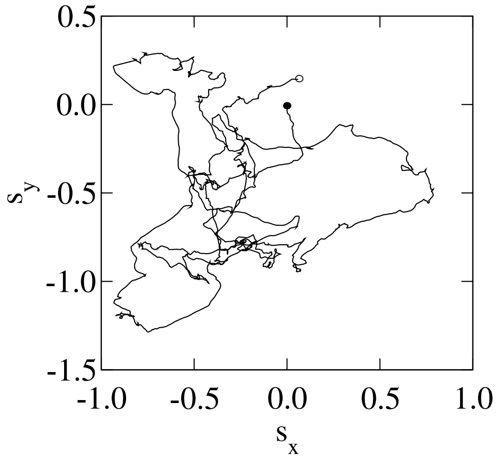

FIG. 5. Diffusive trajectory of one particle with respect to the mean background flow expressed in the unit of mean particle diameter D for a cumulative shear strain of 2. The displacements may be compared with the approximate cell dimensions $60D \times 70D$.

particles are more widely separated, a normal diffusion law is recovered.

In summary, we analyzed fluctuating particle displacements with respect to the background quasistatic shear flow in a model granular medium. These fluctuations were shown to have the following scaling characteristics: (i) the pdf's undergo a transition from stretched exponential to Gaussian as the time lag is increased, (ii) the spatial power spectrum of the velocity field obeys a power law, reflecting long-range correlations and the self-affine nature of the fluctuations, and (iii) the fluctuating displacements have a superdiffusive character. These observations contradict somehow the conventional wisdom which disregards kinematic fluctuations in macroscopic modeling of plastic flow in granular media. Several basic aspects of quasistatic granular flow (elementary representative volumes, mean field approximation, memory effects, and mixing) are thus concerned by these findings [12].

There appears an evident analogy with the scaling features of turbulence that was also discussed throughout this Letter. Of course, this analogy does not imply that a QS granular flow can be considered as turbulent in the standard sense of fluid dynamics. In particular, because of the fundamentally different origin of granular fluctuations, a direct reference to the underlying physics of turbulence can be misleading. But, the observed analogy in terms of scaling characteristics is consistent enough to upgrade kinematic fluctuations in quasistatic granular flows to the rank of a systematic phenomenology which could be coined by the term "granulence" as compared to "turbulence" in fluids.

Interestingly, this analogy works with three-dimensional turbulence, although our data concern a two-dimensional granular flow. The energy cascade in turbulence is governed by inertia, and two- and three-dimensional systems

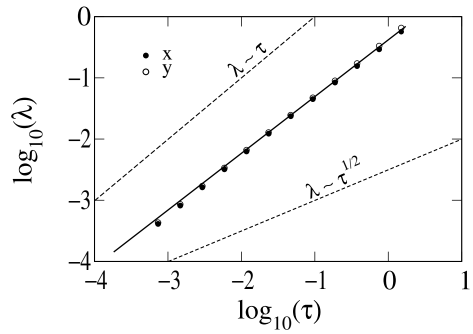

FIG. 6. Root-mean-square relative displacements λ along x and y directions as a function of time τ fitted by a power law of exponent 0.9 (straight solid line). As a guide to the eyes, a power law of exponent $1/2$, corresponding to normal diffusion, and the line $\lambda \propto \tau$ are shown as well (dashed lines).

do differ significantly in this respect. In quasistatic granular flow, the fluctuating velocity field is a consequence of the geometrical compatibility of the strain with particle arrangements, and dissipation is mainly governed by friction at the particle scale. The difference between two- and three-dimensional systems may thus be less crucial, but it was not investigated in the present work. Quite independently of its physical origins, this analogy is suggestive enough to be used as a promising strategy towards a refined probabilistic description of granular flow in the plastic regime.

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