Local and Occupation Time of a Particle Diffusing in a Random Medium

Satya N. Majumdar[1] and Alain Comtet[2,3]

[1]Laboratoire de Physique Quantique (UMR C5626 du CNRS), Université Paul Sabatier, 31062 Toulouse Cedex, France
[2]Laboratoire de Physique Théorique et Modèles Statistiques, Université Paris-Sud, Bâtiment 100, 91405 Orsay Cedex, France
[3]Institut Henri Poincaré, 11 rue Pierre et Marie Curie, 75005 Paris, France

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We consider a particle moving in a one-dimensional potential which has a symmetric deterministic part and a quenched random part. We study analytically the probability distributions of the local time (spent by the particle around its mean value) and the occupation time (spent above its mean value) within an observation time window of size t. In the absence of quenched randomness, these distributions have three typical asymptotic behaviors depending on whether the deterministic potential is unstable, stable, or flat. These asymptotic behaviors are shown to get drastically modified when the random part of the potential is switched on, leading to the loss of self-averaging and wide sample to sample fluctuations.

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How many days, out of a total number of t days, does a tourist spend in a given place? If the tourist is a Brownian particle, this local time (per unit volume) $n_t(\vec{r})$ spent by it in an infinitesimal neighborhood of a point $\vec{r}$ in space has been of interest to physicists and mathematicians for decades. If $\vec{R}(t')$ denotes the position of the random walker at time t', the local time is defined as

$$n_t(\vec{r}) = \int_0^t \delta[\vec{R}(t') - \vec{r}]dt'. \tag{1}$$

Clearly $\int n_t(\vec{r})d\vec{r} = t$. The local time is a very useful quantity with a variety of important applications in fields ranging from physics to biology. For example, in the context of polymers in a solution, the local time $n_t(\vec{r})$ is proportional to the concentration of monomers at $\vec{r}$ in a polymer of length t and can be measured via light scattering experiments. The distribution of local time (LTD) also plays an important role in the study of diffusion in porous rocks [1,2] and in bacterial chemotaxis where the phenomenon of "tumbling" is quantified by the local time spent by a bacteria at a point [3].

A related quantity, also of wide interest to both physicists [4] and mathematicians [5], is the occupation time $T_t(D) = \int_D n_t(\vec{r})d\vec{r}$ spent by the walker in a given region D of space. Recently the study of the occupation time has seen a revival due to its newfound applications in the context of persistence in nonequilibrium statistical physics [6]. The dynamics in such systems is typically modeled by a stochastic process $x(t)$ whose statistical properties provide important information about the history of evolution in these systems [6]. A quantity that acts as a useful probe to this history dependence is the occupation time $T_t = \int_0^t \theta[x(t')]dt'$, the time spent by the process on the positive side within a window of size t [7]. For instance, if $x(t)$ represents a one-dimensional Brownian motion, the probability distribution of the occupation time (OTD) is given by the celebrated "arcsine" law of Lévy [8], $\mathrm{Prob}[T_t = T] = 1/\pi\sqrt{T(t-T)}$ which diverges at the end points $T = 0$ and $T = t$ indicating the "stiffness" of the Brownian motion, i.e., a Brownian path starting at a positive (negative) point tends to remain positive (negative). The OTD has been studied in a variety of physical systems. For example, it has been used to analyze the morphological dynamics of interfaces [9]. Recently the OTD has also been used to analyze the experimental data on the "on-off" fluorescence intermittency emitting from colloidal (CdSe) semiconductor quantum dots [10]. Exotic properties such as a phase transition in the ergodicity of a process by tuning a parameter have also been exhibited by the OTD in the diffusion equation [11].

While the LTD and the OTD have been studied extensively for pure systems, they have so far not been studied in systems with quenched disorder. Given the fact that these distributions have found a host of interesting applications in pure systems as described above, it is natural to expect that they will be equally relevant in disordered systems. In this Letter, we address these questions for the first time in disordered systems. The study of these quantities can provide valuable information in disordered systems. For example, if one launches a tracer particle in a system with localized impurities, the particle diffuses in this random medium, occasionally gets pinned in the region near the impurities till the thermal fluctuations lift it out of the local potential well, and then it diffuses again. Since the local time spent by the particle at a given point in space is related to the concentration of impurities there, it can be used as a valuable probe to image the inhomogeneities in a given sample.

Since nothing is known about the behaviors of the local and the occupation time in disordered systems, the natural first step would be to study them in the simplest possible model of disordered systems. In this Letter we carry out this important first step in such a candidate model—namely, the celebrated Sinai model [12]—where a particle diffuses in a one-dimensional random potential. The Sinai model has long been considered as the "Ising" model of disordered systems that allows the exact calculation of

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various physical quantities [2,13,14] which can then be used to provide the "first guess" for the behaviors of these quantities in more complex realistic disordered systems. Furthermore, we point out that the scope of our results is not limited to be just the first guess. In fact, the Sinai model and its variants have found numerous applications in various physical processes [2,13] including the diffusion of electrons in disordered medium, glassy activated dynamics of dislocations in solids, dynamics of random field magnets, dynamics near the helix-coil transitions in heteropolymers, and more recently the dynamics of denaturation of a single DNA molecule under external force [15]. Therefore our results are expected to be directly relevant for such processes. In particular, motivated by the application in the dynamics of denaturation process [15], we study here a more general version of the Sinai model, where in addition to a random potential, there is also an external deterministic potential whose derivative represents the deterministic force on the particle. Despite the simplicity of the model the LTD and the OTD display a variety of rich and interesting behaviors as shown below.

We start with the Langevin equation of motion of an overdamped particle

$$\frac{dx}{dt} = F(x) + \eta(t), \tag{2}$$

where $\eta(t)$ is a thermal Gaussian white noise with zero mean and a correlator $\langle \eta(t)\eta(t')\rangle = 2k_BT\delta(t-t')$. For simplicity we set $k_BT = 1$. The force $F(x) = -dU/dx$ is derived from a potential which has a deterministic and a random part, $U(x) = U_d(x) + U_r(x)$. In the continuous version of the Sinai model we choose the random potential $U_r(x) = \sqrt{\sigma}\int_0^x \xi(x')dx'$ to be a Brownian motion in space where $\xi(x)$ is a quenched Gaussian noise with zero mean and a correlator $\langle \xi(x)\xi(x')\rangle = \delta(x-x')$. The goal is to first compute the probability distribution of the local time, $n_t(a) = \int_0^t \delta(x(t')-a)dt'$ and that of the occupation time $T_t(a) = \int_0^t \theta(x(t')-a)dt'$ corresponding to level a for a given sample of quenched disorder and then obtain the disorder averaged distributions. For simplicity, we will consider $U_d(x)$ to be symmetric so that the mean position of the particle is at zero and restrict ourselves to study the distributions of $n_t = n_t(0)$ and $T_t = T_t(0)$ corresponding to the natural choice of the level $a = 0$. However, our results are easily generalizable to more general potentials and to arbitrary levels a.

It turns out that the generic asymptotic scaling behaviors of the LTD and the OTD, at a qualitative level, depend on whether the deterministic potential $U_d(x)$ is unstable $[U_d(x) \to -\infty$ as $x \to \pm\infty]$, stable $[U_d(x) \to \infty$ as $x \to \pm\infty]$ or flat $[U_d(x) = 0]$. Quantitatively, however, the LTD and the OTD do depend on the details of the potential $U_d(x)$. To keep the discussion simple, we present explicit results here for the case when $U_d(x) = -\mu|x|$, even though our techniques can be extended to other potentials as well. Thus we will consider Eq. (2) with the force $F(x) = \mu\,\mathrm{sign}(x) + \sqrt{\sigma}\xi(x)$. For $\mu > 0$, the deterministic force is repulsive from the origin and for $\mu < 0$, the force is attractive. For $\mu = 0$, Eq. (2) reduces exactly to the Sinai model.

It is useful to start with a generalized variable $\tau_t = \int_0^x V[x(t')]dt'$ with an arbitrary functional $V[x(t')]$ which reduces to the local time n_t and the occupation time T_t when $V(x) = \delta(x)$ and $V(x) = \theta(x)$, respectively. Let $P_x(T,t)$ be the probability that $\tau_t = T$ given that the particle starts at $x(0) = x$. The Laplace transform $Q_p(x,t) = \int_0^\infty e^{-pT}P_x(T,t)dT$ plays a crucial role in our subsequent analysis. By definition $Q_p(x,t) = \langle e^{-p\int_0^t V[x(t')]dt'}\rangle_x$ where $\langle\rangle_x$ denotes the average over all histories of the particle up to time t starting at x at $t = 0$. Using the evolution Eq. (2) it is straightforward to see that $Q_p(x,t)$ satisfies, for arbitrary $F(x)$, the backward Fokker-Planck equation

$$\frac{\partial Q_p}{\partial t} = \frac{1}{2}\frac{\partial^2 Q_p}{\partial x^2} + F(x)\frac{\partial Q_p}{\partial x} - pV(x)Q_p, \tag{3}$$

with the initial condition $Q_p(x,0) = 1$. After a further Laplace transform, now with respect to t, $u(x) = \int_0^\infty e^{-\alpha t}Q_p(x,t)dt$ satisfies the equation

$$\frac{1}{2}u'' + F(x)u' - [\alpha + pV(x)]u = -1, \tag{4}$$

where $u'(x) = du/dx$ and we have suppressed the α and p dependence of $u(x)$ for notational convenience. So far the discussion is quite general. We now consider the local and the occupation time separately.

Local time.—In this case $V(x) = \delta(x)$. We need to solve Eq. (4) separately for $x > 0$ and $x < 0$ and then match the solutions at $x = 0$. We write $u_\pm(x) = 1/\alpha + A_\pm y_\pm(x)$ where $y_\pm(x)$ satisfy the homogeneous equations

$$\frac{1}{2}y''_\pm + F(x)y'_\pm - \alpha y_\pm = 0, \tag{5}$$

respectively in the regions $x > 0$ and $x < 0$ with the boundary conditions $y_+(x \to \infty) = 0$ and $y_-(x \to -\infty) = 0$. The constants $A_\pm$ are determined from the matching conditions, $u_+(0) = u_-(0) = u(0)$ and $u'_+(0) - u'_-(0) = 2pu(0)$. Eliminating the constants, we get $u(0) = \lambda(\alpha)/\alpha[p + \lambda(\alpha)]$, where $\lambda(\alpha) = [z_-(0) - z_+(0)]/2$ and $z_\pm(x) = y'_\pm(x)/y_\pm(x)$. For simplicity, we will restrict ourselves only to $P_0(n_t = T, t)$ corresponding to the natural choice of the starting point $x(0) = 0$, though our methods can be easily generalized to arbitrary initial positions. Since $\lambda(\alpha)$ is independent of p, one can easily invert the Laplace transform $u(0)$ with respect to p to get

$$G(\alpha) = \int_0^\infty e^{-\alpha t}P_0(n_t = T, t)dt = \frac{\lambda(\alpha)}{\alpha}e^{-\lambda(\alpha)T}, \tag{6}$$

a general result valid for arbitrary $F(x)$. To proceed more we choose $F(x) = \mu\,\mathrm{sign}(x) + \sqrt{\sigma}\xi(x)$ and consider the implications of Eq. (6) for the pure case $\sigma = 0$ first for arbitrary μ.

Solving Eqs. (5) for $F(x) = \mu\mathrm{sign}(x)$ we find $y_{\pm}(x) = y_{\pm}(0)e^{\mp\beta x}$ with $\beta = \mu + \sqrt{\mu^2 + 2\alpha}$. This gives $\lambda(\alpha) = \beta = \mu + \sqrt{\mu^2 + 2\alpha}$. By analyzing this Laplace transform one finds that there are three different asymptotic (large t) behaviors of $P_0(T, t)$ depending on whether the potential is unstable ($\mu > 0$), stable ($\mu < 0$), or flat ($\mu = 0$). These asymptotic behaviors for the particular potential chosen here ($U_d(x) = \mu|x|$) can be shown to be typical for generic potentials.

Unstable potential ($\mu > 0$).—In this case, the LTD approaches a steady state in the large window size $t \to \infty$ limit, $P_0(T) = 2\mu e^{-2\mu T}$ obtained by taking $\alpha \to 0$ limit in Eq. (6). Physically it indicates that for repulsive force the particle eventually goes to either ∞ or $-\infty$ and occasionally hits the origin according to a Poisson process. This asymptotic exponential distribution $P_0(T) = \lambda(0)e^{-\lambda(0)T}$ is indeed universal (up to a rescaling factor of time) for any unstable potential.

Stable potential ($\mu < 0$).—For generic stable potentials the system approaches a stationary state in the large t limit and the stationary probability distribution $p(x)$ for the position of the particle is given by the Gibbs measure, $p(x) = e^{-2U_d(x)}/Z$, where $Z = \int_{-\infty}^{\infty} e^{-2U_d(x)}dx$ is the partition function. Hence as $t \to \infty$, simple ergodicity arguments indicate that the local time $n_t \to \int_0^t \langle\delta[x(t')]\rangle dt' \to p(0)t$, i.e., the LTD is simply $P_0(n_t = T, t) = \delta[T - p(0)t]$, a result that can also be proved rigorously [16]. As an example, for $U_d(x) = -\mu|x|$ considered in this paper, one finds $p(0) = |\mu|$ and hence $P_0(T, t) = \delta(T - |\mu|t)$ as $t \to \infty$.

Flat potential ($\mu = 0$).—In this case, one finds by inverting the Laplace tranform in Eq. (6) that the LTD is Gaussian for all T and t, $P_0(T, t) = \sqrt{2/\pi t}e^{-T^2/2t}$.

These behaviors for the pure system ($\sigma = 0$) get drastically modified when the random potential is switched on ($\sigma > 0$). Equation (6) still remains valid for each realization of $F(x)$. Our aim is to compute the disorder averaged LTD $\overline{P_0(n_t = T, t)}$. From Eq. (6), one needs to know the distribution of $\lambda(\alpha) = [z_-(0) - z_+(0)]/2$ which is now a random variable since $F(x)$ is random. It turns out the distribution of $\lambda(\alpha)$ can be computed exactly by adopting techniques that have appeared before in the Sinai model in other contexts [13,17,18]. We defer the technical details for a future publication [16] and present only the final results here. One gets $\overline{\exp[-\lambda(\alpha)T]} = q^2(T)$ with

$$q(T) = (1 + \sigma T)^{-\mu/2\sigma}\frac{K_{\mu/\sigma}\left(\frac{\sqrt{2\alpha(1+\sigma T)}}{\sigma}\right)}{K_{\mu/\sigma}\left(\frac{\sqrt{2\alpha}}{\sigma}\right)}, \qquad (7)$$

where $K_\nu(x)$ is the modified Bessel function of order ν [19]. Averaging Eq. (6) over disorder we finally get the exact formula

$$\int_0^\infty \overline{P_0(T, t)}e^{-\alpha t}dt = -\frac{1}{\alpha}\frac{d}{dT}[q^2(T)], \qquad (8)$$

where $q(T)$ is given by Eq. (7). The asymptotic behaviors can be deduced by analyzing Eq. (8).

$\mu > 0$: In this case, we find that as $t \to \infty$, $\overline{P_0(T, t)}$ tends to a steady state distribution, $\overline{P_0(T)} = 2\mu(1 + \sigma T)^{-2\mu/\sigma - 1}$ for all $T \geq 0$. Thus the disorder averaged LTD has a broad power law distribution even though for each sample the LTD has a narrow exponential distribution. This indicates wide sample to sample fluctuations and lack of self-averaging.

$\mu < 0$: In this case, for each sample of the disorder, the LTD tends to the delta function $P_0(T, t) \to \delta[T - p(0)t]$ as discussed before. However, the Gibbs measure $p(0)$ varies from sample to sample. On averaging over disorder (or equivalently the peak positions), one finds a broad distribution for $\overline{P_0(T, t)}$. In the scaling limit $t \to \infty$, $n_t = T \to \infty$ but keeping the ratio T/t fixed, we find that $\overline{P_0(T, t)} \to \frac{1}{t}f(T/t)$ where the scaling function can be computed exactly by analyzing Eq. (8),

$$f(y) = \left[\frac{\sqrt{\pi}}{\sigma(2\sigma^3)^{(\nu-1)/2}\Gamma^2(\nu)}\right]y^{3(\nu-1)/2}e^{-y/\sigma}W_{\nu,\nu}(2y/\sigma), \qquad (9)$$

with $\nu = |\mu|/\sigma$ and $W_{\nu,\nu}(x)$ is the Whittaker function [19]. The scaling function increases as $f(y) \sim y^{\nu-1}$ for small y and eventually decays for large y as $f(y) \sim y^{(4\nu-3)/2}e^{-2y/\sigma}$. Once again the disorder modifies the behavior of the LTD rather drastically.

$\mu = 0$ (Sinai model): In this case we find that for large t, $\overline{P_0(n_t = T, t)} \to \frac{1}{t\log^2 t}f_S(T/t)$ where $f_S(y) = 2e^{-y/\sigma}K_0(y/\sigma)/y$. However this scaling breaks down for very small y when $y \ll \sigma$.

We now turn to the OTD, $R_0(T, t) = \mathrm{Prob}(T_t = T, t)$ given that the particle starts at $x = 0$ at $t = 0$. In this case, the double Laplace transform $u_p(x) = \int_0^\infty dte^{-\alpha t}\int_0^t e^{-pT}R_x(T, t)dT$ satisfies Eq. (4) with $V(x) = \theta(x)$. Since the rest of the calculations are very similar to the LTD case, we just present the final results omitting the details.

Unstable potential.—Consider the pure case ($\sigma = 0$) first. Since the deterministic potential is symmetric, one has $R_0(T, t) = R_0(t - T, t)$, i.e., the OTD is symmetric around its mean value $t/2$. In the limit of a large window size $t \to \infty$, it turns out that the part of the OTD to the left of the midpoint $T = t/2$ approaches a steady (t independent) distribution $R_L(T)$ with the normalization $\int_0^\infty R_L(T)dT = 1/2$. The right half of the OTD, which carries an equal total weight $1/2$ is pushed to ∞ since the midpoint $t/2$ itself goes to ∞. This conclusion is valid for any symmetric deterministic potential. For the case $F(x) = \mu\mathrm{sign}(x)$, we get explicitly, $R_L(T) = \mu^2e^{-u^2}[1 - 3\sqrt{\pi}ue^{9u^2}\mathrm{erfc}(3u)]/\sqrt{\pi}u$ where $u = \mu\sqrt{T/2}$ and $\mathrm{erfc}(x)$ is the complementary error function. Thus $R_L(T) \approx \mu\sqrt{2/\pi T}$ for small T and decays exponentially for large T, $R_L(T) \sim T^{-3/2}e^{-\mu^2T/2}$. When the disorder is switched on ($\sigma > 0$), this asymptotic behavior for the pure case does not change qualitatively. Once again the left part of the disordered averaged OTD tends to a t independent form $\overline{R_L(T)}$. In fact, the small T behavior of $\overline{R_L(T)} \approx \mu\sqrt{2/\pi T}$ remains the same as in the pure case. However, for large T, while the OTD still decays exponentially $\overline{R_L(T)} \sim e^{-bT}$, the decay coefficient b turns out to be different from the pure case [16].

Stable potential.—As in the case of the LTD we find that for the pure case, for generic stable potential $U_d(x)$, the OTD approaches a delta function in the $t \to \infty$ limit, $R_0(T_t = T, t) \to \delta(T - \frac{Z_+}{Z} t)$ where Z is the equilibrium partition function and $Z_+ = \int_0^\infty e^{-2U_d(x)} dx$ is the restricted partition function. This result again follows from simple ergodicity arguments. In the presence of disorder ($\sigma > 0$), this asymptotic behavior gets modified as in the case of LTD and we find $\overline{R_0(T, t)} \approx \frac{1}{t} f_o(T/t)$ in the scaling limit. The exact calculation of the scaling function $f_o(T/t)$ is nontrivial but the final answer turns out to be a deceptively simple Beta law,

$$f_o(y) = \frac{1}{B(\nu, \nu)}[y(1-y)]^{\nu-1}, \qquad 0 \le y \le 1, \qquad (10)$$

where $\nu = |\mu|/\sigma$ and $B(\nu, \nu)$ is the standard Beta function [19]. If one tunes the parameter ν by either varying μ or the disorder strength σ, this OTD exhibits an interesting phase transition in the ergodicity of the particle position at $\nu_c = 1$. For $\nu < \nu_c$, the distribution in Eq. (10) is concave with a minimum at $y = 1/2$ and diverges at the two ends $y = 0, 1$. This means that paths with a small number of zero crossings (such that T is close to either 0 or t) carry more weight than the paths that cross many times (for which T is close to $t/2$), i.e., the particle tends to stay on one side of the origin as in the case of a Brownian motion. Exactly the opposite situation occurs for $\nu > \nu_c$ where $f_o(y)$ is maximum at its mean value $1/2$ indicating largest weights for paths that spend equal times on both sides of $x = 0$. It is interesting to notice that similar types of Beta laws also arise in the study of certain perturbed Brownian motion [20].

Flat potential ($\mu = 0$).—For the pure case ($\sigma = 0$), our method reproduces the "arcsine" law for the OTD of an ordinary Brownian motion, $R_0(T, t) = 1/\pi\sqrt{T(t-T)}$. In the presence of disorder ($\sigma > 0$), i.e., for the the Sinai model, we find that the left part of the OTD for $0 \le T \le t/2$ has the large t behavior, $\overline{R_0^L(T, t)} \approx \frac{1}{\log t} R(T)$. The right half of the OTD for $t/2 \le T \le t$ is just the symmetric reflection of the left part. The t independent function $R(T)$ has a complicated form but with simple limiting behaviors, $R(T) \approx \sigma\sqrt{2/\pi T}$ as $T \to 0$ and $R(T) \approx 1/2T$ for large T, consistent with the normalization condition $\int_0^{t/2} \overline{R_0^L(T, t)} dT = 1/2$.

In summary, we have, for the first time, addressed the question of the LTD and the OTD in disordered systems and have obtained exact asymptotic results in the simplest model of disordered systems namely, for a particle moving in a random potential which has a deterministic part as well. Our exact results are consistent with the general notion that "disorder broadens distributions" of physical quantities. Recently several asymptotic exact results for other quantities in the Sinai type models were derived using a real space renormalization group (RG) treatment [14]. Reproducing the exact results presented here either via the RG method or by the replica method and extending our results to higher dimensions and more realistic disordered systems remain as challenging open problems.

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