Testing Statistical Bounds on Entanglement Using Quantum Chaos

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Previous results indicate that while chaos can lead to substantial entropy production, thereby maximizing dynamical entanglement, this still falls short of maximality. Random matrix theory modeling of composite quantum systems, investigated recently, entails a universal distribution of the eigenvalues of the reduced density matrices. We demonstrate that these distributions are realized in quantized chaotic systems by using a model of two coupled and kicked tops. We derive an explicit statistical universal bound on entanglement, which is also valid for the case of unequal dimensionality of the Hilbert spaces involved, and show that this describes well the bounds observed using composite quantized chaotic systems such as coupled tops.

Recently, entanglement has been discussed extensively due to its crucial role in quantum computation and quantum information theory [1]. Since a quantum computer is a many particle system, entanglement is inevitable and even desirable. Entanglement is important both at the hardware and software levels of a quantum computer, as the efficiency of all proposed quantum algorithms is based on it, hence its characterization as a quantum resource. The many particle nature of a quantum computer brings another phenomenon to the fore, that is *chaos*. Some studies have inquired whether chaos will help or hinder in the operation of a quantum computer [2]. At a more basic level several studies have explored the connections between quantum entanglement and classical chaos [3–5], two phenomena that are *prima facie* uniquely quantum and classical, respectively.

Such a connection between entanglement and chaos has been previously studied with the example of an *N*-atom Jaynes-Cummings model [3]. It was found that the entanglement *rate* is considerably enhanced if the initial wave packet was placed in a chaotic region. In another work of a similar kind, the authors have related such rates to classical Lyapunov exponents with the help of a coupled kicked top model [4]. Recently, one of us studied entanglement in coupled standard maps [5] and found that entanglement increased with coupling strength, but after a certain magnitude of coupling strength corresponding to the emergence of complete classical chaos, the entanglement saturated. The saturation value depended on the Hilbert space dimensions and was less than its maximum possible value. This result implies that, though there exists a maximum kinematical limit for entanglement, dynamically it is not possible to create it by using generic Hamiltonian evolutions on unentangled states. It should be emphasized that such bounds are *statistical* and are more unlikely to be violated the larger the Hilbert space dimension.

Recent related work [6] calculates the mean entanglement of pure states for the case $M = N$ by using a random matrix theory (RMT) model that allows specification of the joint probability distribution of the eigenvalues of the reduced density matrices (RDM). Below we calculate the entanglement from an eigenvalue distribution that is valid for large M and N. We show that this distribution describes well those obtained from a coupled kicked top model. There is also some early work that calculates the subsystem entropy for random pure states [7]. Apart from RMT simulations, we deal with an actual quantum mechanical system and relate these results to the presence of classical chaos; thus our results show in what context results such as in [6,7] can be expected to be universal.

The previous studies on entanglement, in the context of chaos, were based on pure states of bipartite systems, where the von Neumann entropy of the RDM is a natural measure of quantum entanglement. We will also initially consider pure states and point out in the end that as a simple corollary we can estimate the entanglement of formation of any density matrix as well. Suppose that the state space of a bipartite quantum system is $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, where $\dim \mathcal{H}_1 = N \leq \dim \mathcal{H}_2 = M$, and $\dim \mathcal{H} = d = NM$. If $\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i|$ is an ensemble representation of an arbitrary state in $\mathcal{H}$, the entanglement of formation is found by minimizing $\sum_i p_i E(|\phi_i\rangle)$ over all possible ensemble realizations. Here E is the von Neumann entropy of the RDM of the state $|\phi_i\rangle$ belonging to the ensemble, i.e., its entanglement. For pure states $|\psi\rangle$ there is only one unique term in the ensemble representation and the entanglement of formation is simply the von Neumann entropy of the RDM.

The two RDMs of the bipartite state $|\psi\rangle$ are $\rho_1 = \text{Tr}_2(|\psi\rangle\langle\psi|)$ and $\rho_2 = \text{Tr}_1(|\psi\rangle\langle\psi|)$. The Schmidt decomposition of $|\psi\rangle$ is the optimal representation in terms of a product basis and is given by

$$|\psi\rangle = \sum_{i=1}^{N} \sqrt{\lambda_i} |\phi_i^{(1)}\rangle |\phi_i^{(2)}\rangle, \qquad (1)$$

where $0 < \lambda_i \leq 1$ are the (nonzero) eigenvalues of either RDMs and the vectors are the corresponding eigenvectors. The von Neumann entropy S_V is the entanglement $E(|\psi\rangle)$ given by

$$S_V = -\mathrm{Tr}_l(\rho_l \ln \rho_l) = -\sum_{i=1}^{N} \lambda_i \ln(\lambda_i); \qquad l = 1, 2. \quad (2)$$

Under an arbitrary unitary evolution ρ or $|\psi\rangle$ evolves into states with changed entanglement. Quantum chaotic evolutions eventually create large entanglement which fluctuates around the value $\ln(\gamma N)$. The factor γ depends only on the ratio M/N and tends to unity (maximal entanglement) as $M \to \infty$. Such evolutions lead to *universal* properties of the RDMs, which is also shared by the RDMs of stationary states. This follows from the universal near normal distributions of the complete pure state components in any generic bases. The distribution of the eigenvalues of RDMs $\{\lambda_i\}$ also follows from RMT results for correlation matrices recently used in the analysis of data from financial time series [8]. Many important universal spectral fluctuation properties of quantum chaotic systems have been modeled and explained by RMT. We extend this success to the RDMs of composite systems and consequently a description of quantum entanglement in strongly interacting systems.

As the Hilbert space dimension and chaos have roles in this bound for entanglement, coupled large spins are attractive models. A coupled kicked tops model has already been used in this context [4]; we generalize it here to include the case of unequal spins and symmetry breaking terms. The Hamiltonian of the coupled top system used is

$$H(t) = \frac{\pi}{2} J_{y_1} + \frac{k}{2j_1}(J_{z_1} + \alpha_1)^2 \sum_{n=-\infty}^{\infty} \delta(t-n) + \frac{\pi}{2} J_{y_2}$$
$$+ \frac{k}{2j_2}(J_{z_2} + \alpha_2)^2 \sum_{n=-\infty}^{\infty} \delta(t-n)$$
$$+ \frac{\epsilon}{\sqrt{j_1 j_2}} J_{z_1} J_{z_2} \sum_{n=-\infty}^{\infty} \delta(t-n). \quad (3)$$

The J_{y_r} terms describe free precession of each top and the remaining terms are due to periodic δ-function kicks. The first two such terms are torsion about the z axis and the final term describes the spin-spin coupling. When either of the constants, α_1 or α_2, is not zero the parity symmetry $RH(t)R^{-1} = H(t)$, where $R = \exp(i\pi J_{y_1}) \otimes \exp(i\pi J_{y_2})$, is broken. The dimensionality of the Hilbert spaces is $N = 2j_1 + 1$ and $M = 2j_2 + 1$. The unitary time evolution operator corresponding to this Hamiltonian is given by

$$U_T = (U_1 \otimes U_2)U_{12}^\epsilon = [(U_1^f U_1^k) \otimes (U_2^f U_2^k)]U_{12}^\epsilon, \quad (4)$$

where the different terms are given by

$$U_i^f \equiv \exp\left(-\frac{i\pi}{2} J_{y_i}\right), \qquad U_i^k \equiv \exp\left(-\frac{ik}{2j_i}(J_{z_i} + \alpha_i)^2\right),$$

$$U_{12}^\epsilon \equiv \exp\left(-\frac{i\epsilon}{\sqrt{j_1 j_2}} J_{z_1} J_{z_2}\right), \quad (5)$$

and $i = 1, 2$. There exists an antiunitary generalized

time reversal symmetry, $[\exp(i\pi J_{x_1}) \exp(i\pi J_{y_1}/2)] \otimes [\exp(i\pi J_{x_2}) \exp(i\pi J_{y_2}/2)]K$, where K is a complex conjugation operator, from which we can expect the applicability of results concerning the Gaussian orthogonal ensemble. We note that for the parameter values considered in this Letter the nearest neighbor spacing distribution (NNSD) of the eigenangles of U_T is Wigner distributed, which is typical of quantized chaotic systems with time reversal symmetry. Entanglement production of time evolving states under U_T has been studied for two different initial states. (i) The initial state is a product of directed angular momentum states as given in Ref. [4], placed in the chaotic sea of phase space. This is a completely unentangled state. (ii) The initial state is maximally entangled and is given by

$$\langle m_1, m_2 | \psi(0) \rangle = \frac{1}{N} \delta_{m_1 m_2}. \quad (6)$$

These initial states are evolved under U_T, and the results are displayed in Fig. 1. Here the coupling strength is very large compared to the value taken in [4], as our goal is to study entanglement saturation and strong coupling will help us achieve entanglement saturation within a short time. In the first case, initially both the von Neumann entropy and the linearized entropy $[S_R = 1 - \mathrm{Tr}_1(\rho_1^2)]$ are zero, but with time evolution both entropies start increasing and get saturated, apart from small fluctuations, at values less than their maximum possible values.

For the von Neumann entropy the saturation value is $\sim \ln(0.6N)$ and for the linear entropy it is approximately $1 - 2/N$, where N is the dimension of each subsystem. This is the dynamical bound for entanglement of a system
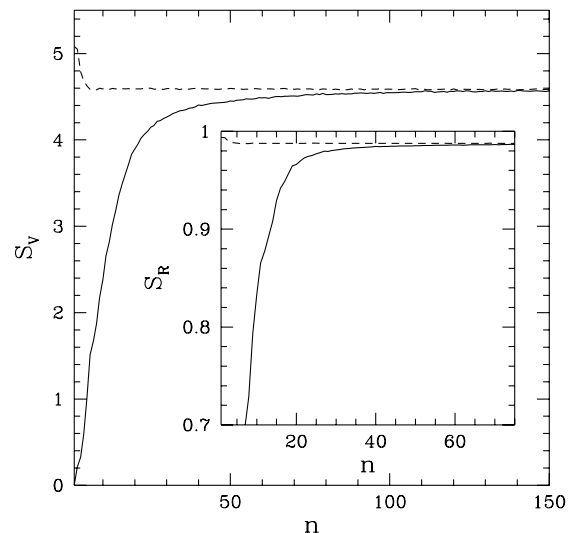

FIG. 1. Entanglement saturation of a completely unentangled initial state (solid line) and a maximally entangled initial state (dotted line) under time evolution operator U_T. Here $k = 3$, $\epsilon = 0.1$ and the phases $\alpha_1 = \alpha_2 = 0.47$. The inset shows the similar behavior of linear entropy.

consisting of two equal dimensional subsystems, while the maximum kinematical limits are $\ln N$ and $1 - 1/N$, respectively. The saturation value of von Neumann entropy of this time evolved state is the same as that obtained for stationary states of completely chaotic coupled standard maps [5]. In the second case, the initial state is maximally entangled and time evolution forces this state to partially disentangle till the entropy reaches the above mentioned values.

This study shows that the saturation of entanglement is a universal phenomenon; it depends only on the Hilbert space dimensions, and not on dynamical characteristics of the system, apart from the presence of complete chaos. The effect of dimension on entanglement saturation has been studied by keeping the dimension of the first subsystem constant at N and increasing the dimension M of the second subspace from $M = N$ to some large value. Thus we may think of the second spin as tending towards a complex bath with a quasicontinuous spectrum. It is observed that the entanglement saturation increases with M and finally gets saturated at the maximum possible kinematical limit, as shown in Fig. 2. For example, the von Neumann entropy starting from $\ln(0.6N)$ increases asymptotically to $\ln N$, while the linear entropy starting from $1 - 2/N$ tends to $1 - 1/N$.

We can develop a complete analytical understanding of these limits via RMT modeling. A typical stationary state
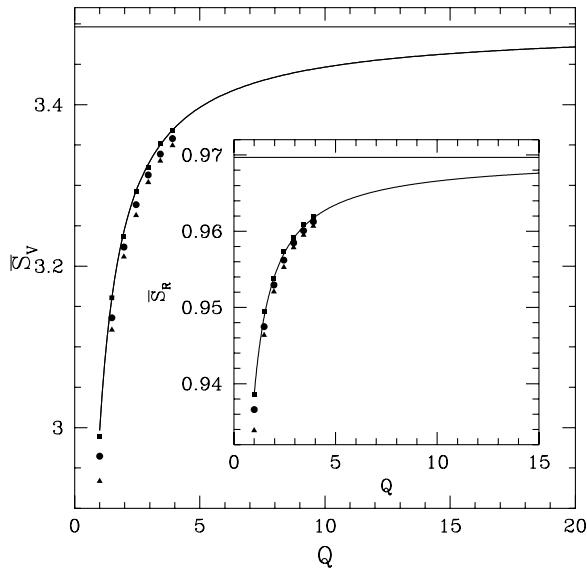

FIG. 2. The spectral average of the entanglement present in eigenstates of U_T ($k = 9$, $\epsilon = 10$) as a function of $Q = M/N$, where $N = 2j_1 + 1 = 33$. Solid triangles are kicked top results with parity symmetry ($\alpha_1 = \alpha_2 = 0$) and solid circles are the corresponding results without symmetry ($\alpha_1 = \alpha_2 = 0.47$). Solid squares are the result of corresponding RMT Monte Carlo simulations and the solid line is the theoretical curve [Eq. (8)]. The horizontal line is the maximum possible entanglement [$\ln(N)$]. The inset shows the behavior of the linear entropy.

of a quantum chaotic system shares properties of the eigenvectors of random matrices. Let us assume that some product basis has been used to write components a_{nm} of any state, which is real for stationary states of time reversal symmetric systems. Writing a_{nm} as the rectangular $N \times M$ matrix A, the M dimensional RDM is $A^T A$, while the other RDM is the N dimensional AA^T. The assumptions of quantum chaos, as we have just seen, imply that A can be taken to have random independent entries, a member of the Laguerre ensemble. The RDMs then have the structure of correlation matrices [8], from where we directly use results for the density of states. Such matrices have also been studied since the early days of RMT as they have a non-negative spectrum [9]. The distribution of the eigenvalues of such matrices is known and thus this is the distribution of the eigenvalues of RDMs. The density of the eigenvalues of the RDM ρ_1 is given by

$$
f(\lambda) = \frac{NQ}{2\pi} \frac{\sqrt{(\lambda_{\max} - \lambda)(\lambda - \lambda_{\min})}}{\lambda}
$$
$$
\lambda_{\min}^{\max} = \frac{1}{N}\left(1 + \frac{1}{Q} \pm \frac{2}{\sqrt{Q}}\right),
\tag{7}
$$

where $\lambda \in [\lambda_{\min}, \lambda_{\max}]$, $Q = M/N$, and $Nf(\lambda)d\lambda$ is the number of eigenvalues within λ to $\lambda + d\lambda$. This has been derived under the assumption that both M and N are large. Note that this predicts a range of eigenvalues for the RDMs that are of the order of $1/N$. For $Q \neq 1$, the eigenvalues of the RDMs are bounded away from the origin, while for $Q = 1$ there is a divergence at the origin. All of these predictions are seen to be borne out in numerical work with coupled tops.

Figure 3 shows how well the above formula fits the eigenvalue distribution of actual reduced density matrices. This figure also shows that the probability of getting an eigenvalue outside the range $[\lambda_{\min}, \lambda_{\max}]$ is indeed very small. The sum in S_V can be replaced by an integral over the density $f(\lambda)$:

$$
S_V \sim - \int_{\lambda_{\min}}^{\lambda_{\max}} f(\lambda)\lambda \ln\lambda \, d\lambda \equiv \ln(\gamma N).
\tag{8}
$$

The integral in γ can be evaluated to a generalized hypergeometric function and the final result is

$$
\gamma = \frac{Q}{Q+1}\exp\left[\frac{Q}{2(Q+1)^2} {}_3F_2\left\{1, 1, \frac{3}{2}; 2, 3; \frac{4Q}{(Q+1)^2}\right\}\right].
\tag{9}
$$

When the two subsystems are of equal dimension, that is, $Q = 1$, then the above expression gives $\gamma = \exp(-0.5) \sim 0.6$ and so the corresponding von Neumann entropy is $\ln(0.6N)$. This is also the saturation value obtained in previous numerical work for the stationary states and time evolving states of a coupled standard map [5], reflecting universality. In another extreme case, when the Hilbert space dimension of the second subsystem is very large
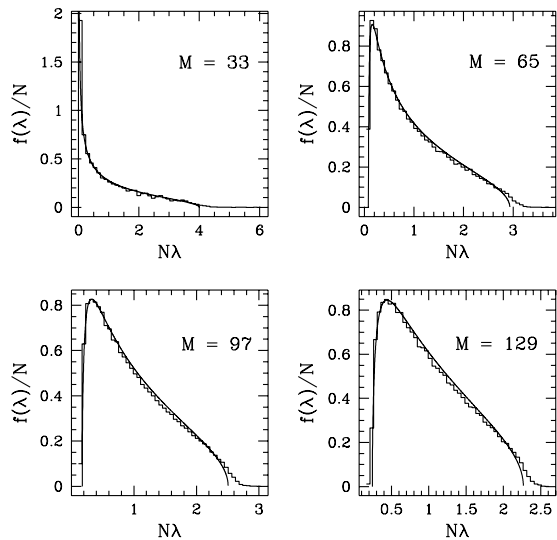
FIG. 3. Distribution of the eigenvalues of the RDMs of coupled kicked tops, averaged over all the eigenstates ($N = 2j_1 + 1 = 33$). Solid curves correspond to the theoretical distribution function [Eq. (7)].

compared to that of the first, that is, $Q \gg 1$, then $\gamma \sim 1$ and hence the corresponding von Neumann entropy is $\ln(N)$. Therefore, the analytical formulation based on RMT is able to explain the saturation behavior of the von Neumann entropy or quantum entanglement very accurately.

Figure 2 also compares Eq. (8) to both RMT simulations and kicked top results. We expect that the deviations of the quantum calculations are due to finite size effects. The presence of parity symmetry results in a somewhat smaller entanglement, as seen in this figure, a fact that needs further study. For time evolving states and stationary states of system without time reversal symmetry the RDMs are complex Hermitian matrices. The entanglement bounds discussed here are also valid for these cases as the entanglement depends only on the density of states of the RDMs. However, spectral fluctuations of the RDMs (such as *their* NNSD) corresponding to these states can be distinct. Indeed, in the correlation matrix approach to atmospheric data, such a difference has been recently noted [10].

The linear entropy can also be derived as above, but using direct RMT results, without taking recourse to the distribution above, is also possible in this case. Thus we may write

$$\mathrm{Tr}\rho^2 = \sum_{j,k=1}^{N} \sum_{\alpha,\beta=1}^{M} a_{j\alpha} a_{k\alpha} a_{k\beta} a_{j\beta}. \tag{10}$$

Substituting RMT ensemble average values [11] of $\langle a_{j\alpha}^2 a_{j\beta}^2 \rangle = \langle a_{j\alpha}^2 a_{k\alpha}^2 \rangle = 1/[MN(MN+2)]$, $\langle a_{j\alpha}^4 \rangle = 3/[MN(MN+2)]$, and $\langle a_{j\alpha} a_{k\alpha} a_{k\beta} a_{j\beta} \rangle = 0$, where $j \neq k$ and $\alpha \neq \beta$ in the above expressions, we find that

$$S_R = 1 - \mathrm{Tr}\rho_1^2 = 1 - \frac{M+N+1}{MN+2}. \tag{11}$$

When the dimensions of the two subsystems are equal, that is, $M = N$, then in the large N limit $S_R \sim 1 - 2/N$. This is the saturation value of the linear entropy approximately obtained in the case of time evolving states of coupled kicked tops. Similarly, when the Hilbert space dimension of the second subsystem is very large compared to the dimension of the first subsystem, that is, $M \gg N$, $S_R \sim 1 - 1/N$. This is the maximum possible value of linear entropy.

Finally as an almost trivial corollary we note that the entanglement of formation of a time evolving density matrix is also statistically bounded to $\ln(\gamma N)$, as each pure state belonging to an ensemble representation evolves to this entanglement under quantum chaos. To summarize, we have pointed out that the eigenvalue distribution of reduced density matrices of composite quantum chaotic bipartite systems is universal, and we have shown that there exists a typical value of quantum entanglement that quantum chaos engenders. This value is the maximum we may expect typical unentangled initial states to be able to reach under generic interactions. If we already had maximally entangled states, then chaos can disentangle this state to just such an extent as to coincide with this generic value.

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