Freeze-Out Problem in Hydrokinetic Approach to $A + A$ Collisions

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A new method for evaluating spectra and correlations in the hydrodynamic approach is proposed. It is based on an analysis of the Boltzmann equations (BE) in terms of probabilities for constituent particles to escape from the interacting system. The conditions of applicability of the Cooper-Frye freeze-out prescription are considered within the method. The results are illustrated with a nonrelativistic exact solution of BE for an expanding spherical fireball as well as with approximate solutions for ellipsoidally expanding ones.

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Introduction.—The hydrodynamic approach to multiparticle production in hadron collisions was proposed in 1953 by Landau [1]. It considers as the initial state a very hot and dense thermal gas of particles soon after the collision, which then expands hydrodynamically until some finite time when the picture of continuous medium is destroyed. The latter stage, so-called freeze-out, happens when the mean free path of particles becomes comparable with the smallest of the system's characteristic dimensions: its geometrical size or hydrodynamic length. Hydrodynamic models find serious utilization in the description of high-energy nuclear collisions, especially at CERN SPS and Brookhaven RHIC. Studying predicted spectra of different particle species versus the initial conditions (IC) and equation of state (EoS), one could get information on the early (partonic) stage of the "little bang," such as a possible formation of quark-gluon plasma (QGP) or even a type of the phase transition between QGP and the hadron matter. The problem is, however, whether the predicted spectra with given IC and EoS in hydrodynamic models are unambiguous.

The standard and widely used method to get the spectra is the so-called Cooper-Frye prescription (CFp) [2] that treats the system at the decay stage of evolution as a locally equilibrated ideal gas at some hypersurface. However, this prescription presents some serious problems because, usually, the freeze-out hypersurface contains non-space-like sectors, and, as a result, artificial discontinuities (shock waves) across those hypersurfaces appear to adjust CFp to energy and momentum conservation laws [3]. Moreover, the results of many studies based on cascade models contradict the idea of sudden freeze-out: particles escape from the system during almost the whole time of its evolution and do not demonstrate the local equilibration at late stages. The method of continuous emission [4] gave an important step in the description of particle freeze-out from 4-volume within the hydrodynamic approach. However, it is not fully satisfactory because it is not based on Boltzmann equations (BE) and, as a consequence, when applied to realistic systems, fails as the particle escape probability becomes large. In the present paper we propose an approximate method for describing spectra within the hydrokinetic approach that is simpler for dense systems than a pure microscopic approach and apply it zto a description of particle production from an expanding and interacting Boltzmann gas that is initially in local equilibrium.

Boltzmann equation, probability to escape, and escaping function.—The BE for the distribution function $f(x, p)$ in the case of no external forces has the form

$$\frac{p^{\mu}}{p^{0}}\frac{\partial f(x,p)}{\partial x^{\mu}} = F^{\text{gain}}(x,p) - F^{\text{loss}}(x,p). \tag{1}$$

The terms F^{gain} and F^{loss} are associated with the number of particles which, respectively, came to the point (x, p) and leave this point because of collisions. The term $F^{\text{loss}}(x,p) = R(x,p)f(x,p)$ can easily be expressed in terms of the rate of collisions of the particle with momentum p, $R(x,p) = \langle\sigma v_{\text{rel}}\rangle n(x)$. The term F^{gain} has a more complicated integral structure and depends on the differential cross section. To develop a corresponding method, let us split the distribution function at each space-time point into two parts: $f(x,p) = f_{\text{int}}(x,p) + f_{\text{esc}}(x,p)$, $x = (t, \mathbf{x})$. The first one, $f_{\text{int}}(x,p)$, describes the fraction of the system which will continue to interact after the time t. The second one, $f_{\text{esc}}(x,p)$, describes the particles that will never interact after the time t. Of course, the latter expresses only probabilities and cannot be ignored when one is solving BE: possible interactions with the escaping part also have to be taken into account, in order to find the escape probabilities.

The function $f_{\text{esc}}(x,p)$ is built up as follows. Let us denote by $x' \equiv (t', \mathbf{x} + (\mathbf{p}/p_0)(t'-t))$ the space-time point where a particle in x with momentum p would be if it moved freely. Consider, at each vicinity of the phase-space point (x, p), the number of particles that have escaped from the interacting system during the time interval $(t', t'+dt')$. This *additional* portion of escaped particles can be produced only from the interacting part of the system. Also, these particles are only among those that came to the phase-space vicinity of the point (x', p) *just after* an interaction during the time dt', suffering the last collision there. Therefore, the additional contribution to $f_{esc}(x, p)$ from the time interval $(t', t' + dt')$ is $\mathcal{P}(x', p)F^{gain}(x', p)dt'$ for $t' < t$ and is zero for $t' > t$. Here $\mathcal{P}(x, p)$ is the probability for any *given* particle at x with momentum p not to interact any more, propagating freely. So collecting all the contributions starting from some initial time t_0, we have

$$f_{esc}(x, p) = f_{esc}(x_0, p) + \int_{t_0}^{t} dt' \, \mathcal{P}(x', p)F^{gain}(x', p), \quad (2)$$

where $x_0 \equiv [t_0, \mathbf{x} + (\mathbf{p}/p_0)(t_0 - t)]$ and $f_{esc}(x_0, p)$ corresponds to the portion of the system, which is already free at t_0. It follows from (2) that

$$\frac{1}{\mathcal{P}(x, p)} \frac{p^\mu}{p^0} \frac{\partial}{\partial x^\mu} f_{esc}(x, p) = F^{gain}(x, p). \quad (3)$$

The escape probability $\mathcal{P}(x, p)$ can be expressed explicitly in terms of the rate of collisions along the world line of the free particle with momentum p as

$$\mathcal{P}(x, p) = \exp\left(-\int_t^\infty dt' \, R(x', p)\right) \quad (4)$$

or it satisfies the differential equation

$$\frac{1}{\mathcal{P}(x, p)} \frac{p^\mu}{p^0} \frac{\partial}{\partial x^\mu} \mathcal{P}(x, p) = R(x, p) = \frac{F^{loss}(x, p)}{f(x, p)}. \quad (5)$$

On the other hand, according to the probability definition,

$$f_{esc}(x, p) = \mathcal{P}(x, p)f(x, p). \quad (6)$$

This equation, where f_{esc} is given by Eq. (2) and $\mathcal{P}$ by Eq. (4), is one of the integral forms of BE (1). One can check directly that $f = f_{esc}/\mathcal{P}$ is governed by BE and, thus, our definitions of $f_{esc}(x, p)$ and $\mathcal{P}(x, p)$ are consistent with BE.

For an initially finite system with a short-range interaction among particles, actually the system becomes free at large enough times t_{out}, so $\mathcal{P}(x, p) \to 1$ and $f_{esc}(x, p) \to f(x, p)$ in this limit. Our proposal is to utilize, for the description of particle spectra in $A + A$ collisions, the escaping function (2) with $\mathcal{P}(x, p)$ [and thus $R(x, p)$; see Eq. (4)] and F^{gain} evaluated just in the local equilibrium (l.eq.) approximation for $f(x, p)$. In this case the function $f = f_{esc}/\mathcal{P}$ corresponds to a solution of the kinetic equation in relaxation time approximation,

$$\frac{p^\mu}{p^0} \frac{\partial f(x, p)}{\partial x^\mu} = -\frac{f(x, p) - f_{l.eq.}(x, p)}{\tau(x, p)}, \quad (7)$$

with the relaxation time $\tau(x, p)$ being expressed through the rate of collisions $R_{l.eq.}(x, p)$ of a particle with momentum p, $\tau = 1/R_{l.eq.}$. When $t \to \infty$, the relaxation time $\tau \to \infty$ also, indicating a transition to the free streaming regime. Equation (7) can be derived also from BE with an approximation of the right-hand side (rhs) by $F^{gain} = R_{l.eq.}f_{l.eq.}$ and $F^{loss} = R_{l.eq.}f$ [5]. Note that the macroscopic parameters of the l.eq. distribution function are governed, in general, by nonideal hydrodynamic equations that are derived from Eq. (7) in the standard way (see, e.g., [5]).

The approximation proposed is based on collecting all the *liberated* particles during the system evolution and, therefore, takes into account rather nonequilibrium processes, similar to particle emission from the periphery of a system. Obviously, it should describe well the spectra in a fast transition of the system from an initial equilibrium state to free streaming (e.g., explosion), as well as in a fairly smooth transition when l.eq. is nearly maintained till rather low densities. Formally, comparing with the exact solution of BE, we neglect in Eq. (2) the term

$$\mathcal{P}F^{gain} - \mathcal{P}_{l.eq.}R_{l.eq.}f_{l.eq.}, \quad (8)$$

integrated along the world line of a particle from t_0 to t_{out}. At part of a trajectory crossing a sufficiently dense space-time region, neither F^{gain} nor $\mathcal{P}$ differ much from the corresponding hydro terms in Eq. (8). At the later stage of expansion, when the densities are small [normally $n(x) \sim t^{-3}$], the values of F^{gain} terms, for both hydro and exact solutions, are rather small. So the integrated contribution of the neglected term, Eq. (8), coming from that part of a trajectory, is also small no matter how much the l.eq. distribution function $f_{l.eq.}$ deviates there from the exact solution f. As for the most delicate transition region, one can expect that the neglected term (8) in this region could not be important, too. In detail, the transition from the initial l.eq. state to free streaming is rather fast at a periphery of the system, so the correspondent integrated contribution of (8) is small. The contribution of the neglected term from trajectories crossing a central part of the system could be, for l.eq. initial distributions, small also if the deviation from l.eq. is slight until rather small densities. We shall discuss below some examples of such a behavior.

Particle spectra and correlations.—To describe the inclusive spectra of particles,

$$p^0 \frac{dN}{d\mathbf{p}} = \langle a_p^+ a_p \rangle, \qquad p_1^0 p_2^0 \frac{dN}{d\mathbf{p}_1 d\mathbf{p}_2} = \langle a_{p_1}^+ a_{p_2}^+ a_{p_1} a_{p_2} \rangle, \quad (9)$$

the asymptotic equality $f_{esc}(x, p) = f(x, p)$ can be used, replacing the total distribution function f in all irreducible averages in (9),

$$\langle a_{p_1}^+ a_{p_2} \rangle = \int_{\sigma_{out}} d\sigma_\mu \, p^\mu \exp(iqx)f(x, p), \quad (10)$$

by f_{esc}. Here $p = (p_1 + p_2)/2$, $q = p_1 - p_2$, and the hypersurface σ_{out} just generalizes t_{out}. Applying the Gauss theorem and recalling that $\partial_\mu[p^\mu \exp(iqx)] = 0$

for particles on the mass shell, one obtains, using, respectively, general Eqs. (3) and (1) and supposing their analytical continuation off the mass shell,

$$\langle a^+_{p_1} a_{p_2} \rangle = p^\mu \int_{\sigma_0} d\sigma_\mu f_{\rm esc}(x_0, p) e^{iqx} + p^0 \int_{\sigma_0}^{\sigma_{\rm out}} d^4x \mathcal{P}(x, p) F^{\rm gain}(x, p) e^{iqx}, \quad (11)$$

$$\langle a^+_{p_1} a_{p_2} \rangle = p^\mu \int_{\sigma_0} d\sigma_\mu f(x_0, p) e^{iqx} + p^0 \int_{\sigma_0}^{\sigma_{\rm out}} d^4x [F^{\rm gain}(x, p) - F^{\rm loss}(x, p)] e^{iqx}. \quad (12)$$

Thus, the use of an escaping function as the asymptotic interpolation to the solution of BE is equivalent to taking, as the source function for the spectra and correlations, the 4-volume emission function $S = \mathcal{P}F^{\rm gain}$ together with direct emission $f_{\rm esc}(x_0, p)$ from an initial 3D hypersurface σ_0. The CFp, defined by Eq. (10) with substitution $\sigma_{\rm out} \to \sigma_{\rm f.o.}$ and $f \to f_{\rm l.eq.}$, treats particle spectra as results of rapid *conversion* of a l.eq. hadron system into a gas of free particles at some hypersurface $\sigma_{\rm f.o.}$. Formally, it corresponds to taking the cross section tending to infinity at $t < t_{\sigma_{\rm f.o.}}$ (to keep a system in the l.eq. state) and zero beyond $t_{\sigma_{\rm f.o.}}$. Then $\mathcal{P}(t, \mathbf{x}, p) = \theta[t - t_{\sigma_{\rm f.o.}}(\mathbf{x})]$ (and so $f_{\rm esc} = 0$ at $t < t_{\sigma_{\rm f.o.}}$), and $S = \mathcal{P}F^{\rm gain} = \delta[t - t_{\sigma_{\rm f.o.}}(\mathbf{x})] f_{\rm l.eq.}$ in Eq. (11).

Let us now analyze the particle-spectrum formation process based on simple analytical models. It was recently shown [6] that, if at an initial time $t = t_0 = 0$ a nonrelativistic ideal gas has an ellipsoidally symmetric Gaussian density distribution and a self-similar velocity profile $\mathbf{u}(x)$, then the solution for ideal hydrodynamics has the form $[\mathbf{C} = \mathbf{v} - \mathbf{u}(x), \mathbf{v} = \mathbf{p}/m, V = \prod_{i=1}^3 X_i]$

$$f(t, \mathbf{x}, \mathbf{v}) = \frac{N}{V} \left(\frac{m}{(2\pi)^2 T} \right)^{3/2} \exp\left(-\frac{m\mathbf{C}^2}{2T} - \sum_{i=1}^3 \frac{x_i^2}{2X_i^2} \right), \quad (13)$$

where

$$X_i \ddot{X}_i = \frac{T}{m}, \qquad T = T_0 \left(\frac{V_0}{V} \right)^{2/3}, \qquad u_i = \frac{\dot{X}_i}{X} x_i. \quad (14)$$

In the spherically symmetric case, when all X_i are equal, the l.eq. distribution function (13) with (14) is an exact solution of BE [7]. One can easily check that the momentum spectrum as well as the interferometry radius, computed at any time t (at any isotherm) or at any other hypersurface enclosing the system, are identical to those calculated at the initial time $t = 0$. The reason is that, in the spherically symmetric case, the l.eq. function (13) makes the left-hand side of BE (1) zero, as well as the rhs, rendering the evolution of the *interacting* gas similar to a free streaming. Then the volume integral in Eq. (12) vanishes, and in this case CFp formally gives the correct spectrum, but its physical meaning is completely different from the naively expected one. There is no clearly defined unique freeze-out hypersurface: neither the escape probability $\mathcal{P}$ nor the emission function S reveals a sharp behavior in the $(t, \mathbf{x})$ plane, but are rather smooth functions. This is demonstrated in Fig. 1 with the average (in $\mathbf{v}$) emission function $\langle S \rangle = \langle \mathcal{P}F^{\rm gain} \rangle = \langle \mathcal{P}Rf \rangle$, where $\mathcal{P}$ is calculated according to Eq. (4).

Therefore, we can conclude that the difference between final spectra (and interferometry radii) at $\sigma_{\rm out}$ and what could be found at σ_0 is due to dissipative effects (deviations from l.eq.), which make the integral over 4-volume in Eq. (12) nonvanishing. The contribution of dissipative effects can be essential even if the evolution of the system is governed with good accuracy by ideal fluid hydrodynamics that take place for fairly high densities or/and cross sections, and so $f(x, p) \approx f_{\rm l.eq.}(x, p)$. In this case, $p^\mu \partial_\mu f(x, p) \approx p^\mu \partial_\mu f_{\rm l.eq.}(x, p) \sim \kappa f_{\rm l.eq.}(x, p)$, where κ does not depend on the density and the cross section but is tied, in particular, with symmetry properties of the hydrodynamic expansion. For the above discussed exact spherically symmetric solution of BE, $\kappa = 0$, but in the general case $\kappa \neq 0$ and, as a result, for high densities particle rescattering can lead to a serious changing of momentum spectra as compared to the initial ones, depending on the initial conditions.

Typically, however, the hadron system is not at local equilibrium during a fairly long later stage of evolution and f could largely deviate from $f_{\rm l.eq.}$. The proposed method of escape probabilities with l.eq. approximation for $\mathcal{P}(x, p)$ and $F^{\rm gain}$ allows one to use the hydrodynamic approach for calculations of spectra even in the situation when finite inhomogeneous systems go through all stages: from local equilibrium to free streaming. Let us illustrate this, based

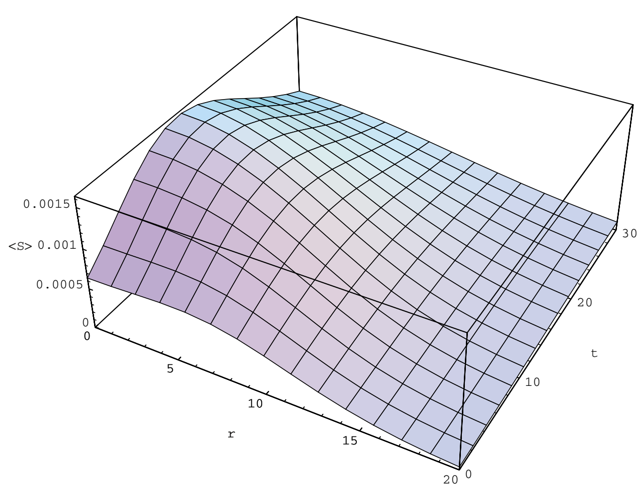

FIG. 1 (color online). The space-time (t, r) dependence of the emission function averaged over momenta for an expanding spherically symmetric fireball containing 400 particles with mass 1 GeV and with cross section $\sigma = 40$ mb, initially at rest and localized with Gaussian radius parameter $R = 7$ fm and temperature $T_0 = 0.130$ GeV.

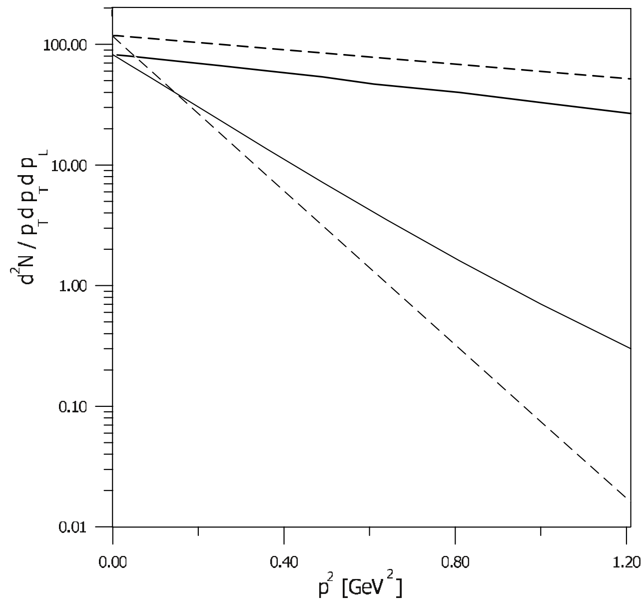

FIG. 2. The transverse (bottom, at $p_L = 0$) and longitudinal (top, at $p_T = 0$) spectra of particles, escaped until $t = 8$ fm$/c$ from an expanding ellipsoidally symmetric fireball of the same particles as in Fig. 1, initially at rest and localized with Gaussian radius parameters $X_1 = X_2 = 7$ fm, $X_3 = 0.7$ fm, and temperature $T_0 = 0.300$ GeV. Dashed lines correspond to spectra calculated according to CFp, applied to the l.eq. distribution function at $T(t) = 0.063$ GeV.

on the exact solution (14) of ideal hydrodynamics for an initially compressed ellipsoid in the longitudinal direction, containing the same gas as in the previous example. The results are presented in Fig. 2, where single-particle spectra at $t = 8$ fm for particles that "escaped" [became free according to Eq. (2)] up to this moment are shown versus those corresponding to frozen-out ones according to CFp, applied to l.eq. distribution function (13) on the isotherm $T(t)$. One can see that the effective temperature of the transverse spectrum calculated by using $f_{\rm esc}(x, p)$ is higher than the one given by the l.eq. distribution function. This is in agreement with the results of partonic cascade algorithm of Ref. [8] and also with those of continuous emission [4]. The longitudinal spectrum shows an opposite tendency.

The solution of the ideal hydrodynamics (13) and (14) also shows that $(\dot{X}_i/X_i)/(\dot{X}_j/X_j) \rightarrow 1$ with time increasing, so the velocity field of the expanding system tends to a spherically symmetric one. Such a tendency of the velocity field is preserved, in the central region, also by the solutions of the Navier-Stokes equation with the same initial conditions, if transport coefficients are calculated according to the Chapman-Enskog (CE) method (we used a hard-sphere model of interaction). Because of this, the deviation from l.eq. in the central part of the system, calculated within the CE method, is rather negligible until the density becomes quite small. Then, there is a hope to apply CFp for the soft-momentum spectra, since these particles are mainly emitted from there. As for the hard-momentum spectra, one expects that such particles are mostly radiated from the periphery of the system at the early times, because of large hydrodynamic velocities and fast transition to free streaming there. So, in a rough approximation, one can apply the generalized CFp, taking into account, first, p-dependent hypersurface $\sigma_{\rm f.o.}$ and, second, the deviations on $\sigma_{\rm f.o.}$ of the distribution function from l.eq. due to dissipative effects.

Conclusion.—The proposed method allows one to describe, in a hydrodynamic approach for $A + A$ collisions, the evolution of the matter from l.eq. till free streaming. The method differs from the continuous emission developed in Refs. [4], where the central object is the interacting component $f_{\rm int}$ which is approximated by the l.eq. distribution function. However, it is possible to show, by using an exact solution of BE, that $f_{\rm int}$ has neither isotropic nor thermal distribution in its local rest frame, even at a moderate value of $\langle \mathcal{P}(x, p) \rangle$.

It is worth noting that our method, applied to the evolution of rarefied hadron gas, overlaps with transport models. In this aspect, the exact solution of BE discussed above is useful as a test of numerical cascade algorithms. The advantage of the method of escaped particles reveals that when the system is not dilute, it has more complicated interactions and displays a collective behavior. The hydrodynamics is still (or even more) suitable for the description of such a system, and so our method based on calculations of probabilities for constituent particles to escape can be used, considering a concrete model for the particle interaction with surrounding medium.

The analysis of the particle-liberation process demonstrates the crucial role of dissipative effects in the formation of one- and two-particle spectra. These effects should be taken into account when experimental data from SPS and RHIC are treated in a hydrodynamic approach.

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