Dilute Quantum Droplets

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In the limit when the two-body scattering length a is negative and much larger than the effective two-body interaction radius the contribution to the ground state energy due to the three-body correlations is given by the Efimov effect. For particular values of the diluteness parameter $\rho|a|^3$ the three-body contribution can become the dominant term of the energy density functional. Under these conditions both Bose-Einstein and Fermi-Dirac systems could become self-bound and either boson droplets or fermion "designer nuclei" of various sizes and densities could be manufactured.

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The extraordinary progress in realizing Bose-Einstein condensates (BEC) and of the Fermi-Dirac counterparts in traps [1,2] raises the question of whether they can also exist in isolation, without a trap potential. In the dilute limit, the physical properties are entirely determined by the two-particle scattering length a. To leading order in a the energy per particle of a uniform condensate is given by

$$E_2 = \frac{1}{2} g_2 \rho, \qquad g_2 = \frac{4\pi\hbar^2 a}{m}, \tag{1}$$

where ρ is the number density and m is the particle mass. For positive a the energy is positive, suggesting that there is no binding. The energy density, including higher order corrections, is given by

$$\mathcal{E}_2(\rho) = \frac{g_2\rho^2}{2}\left\{1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + \left[\frac{8(4\pi - 3\sqrt{3})}{3}\ln(\rho a^3) + C\right]\rho a^3 + \ldots\right\}, \tag{2}$$

where C is a constant [3–5]. These higher order corrections depend on dimensionless parameter ρa^3 up to the term C, which depends on the range of the two-particle interaction. A similar expression to Eq. (2), but for the energy of a dilute Fermi system, in the case of positive scattering lengths, was derived essentially in parallel with the Bose case [3,4]. The positive energy state is stable with respect to long wave density fluctuations, but is metastable only if the underlying two-particle interaction supports bound states. In fact, most of the BEC decay into the two-particle bound states, but they are still quite long lived. The decay rate of the condensate depends on a as $\Gamma \propto a^4\rho^2$; see Refs. [6] and earlier references cited therein for details and some rather unexpected features. In contrast, when the effective scattering length is attractive ($a < 0$), the energy expression Eq. (1) is unbounded from below and the uniform density state is unstable with respect to density fluctuations. The time scale here is very short ($\Gamma \propto \sqrt{|a|\rho}$). It would thus seem that a droplet of BE condensed matter either would be unbound or would decay very quickly. This is not necessarily the case. The picture is radically changed when one considers the effects of the three-body correlations.

Recently it was confirmed experimentally that one can now control the two-body scattering length by placing the atoms in an external field [7]; see also Refs. [8]. For those cases where successful Bose-Einstein condensation was achieved, the bare or unadulterated by any external fields two-body scattering length a was comparable in magnitude with the two-body potential radius. More exactly, the value of the scattering length is comparable with the van der Waals length [5,9]. A new regime is now attainable, when the Bose system is still dilute, but with respect to a new characteristic length. If in the two-body system the scattering length is very large, i.e., if $|a| \gg r_0$, there are now two independent dimensionless parameters, ρa^3 and ρr_0^3. In this limit in the three-body system one has a very unusual phenomenon, the so-called Efimov effect [10], which is manifested in the appearance of a very large number of three-body bound states $N \approx s_0/\pi \ \ln(|a|/r_0)$, where $s_0 \approx 1.0064$, all with the same quantum numbers 0^+. The spatial extensions of these Efimov states range from distances of the order of $O(r_0)$, for the tightest bound one, to distances $O(a)$, for the least bound state. These three-body bound states appear irrespective of whether the two-body scattering length is positive or negative. The sizes of these states change from one state to the next by a factor $\exp(\pi/s_0) \approx 22.68$, while their energies change by a factor $\exp(-2\pi/s_0)$. The spectrum has an exponential character and the bound state wave functions obey a simple scaling law. The properties of these states easily follow from the fact that in the region $r_0 \ll R \ll |a|$, where $R^2 = 2(r_{12}^2 + r_{23}^2 + r_{31}^2)/3$ and r_{kl} is the distance between particles k and l $(k, l = 1, 2, 3)$, there is an effective three-particle interaction with a universal attractive character $-s_0^2\hbar^2/2mR^2$. The universality of this effect resides in the fact that all its properties are fully determined by the

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scattering length a, the radius of the two-body interaction r_0, the logarithmic derivative Λ of the three-body wave function at $R \approx r_0$, and the dimensionless constant s_0. If the particles have spin, the situation can become a little bit more complicated, as the Efimov states can appear for various values of the total spin of the three-body system. Typically the spectrum ceases to be strictly exponential and the wave functions have a somewhat more complex structure. However, on average the qualitative features of these new type of Efimov three-body states remain largely unchanged [11].

Efimov [12] has also shown that at zero energy the amplitude for the three-particle collisions is determined by the same universal interaction $-s_0^2\hbar^2/2mR^2$. In the case of negative scattering lengths ($a < 0$) this amplitude is given by the following universal formula:

$$g_3 = \frac{12\pi\hbar^2 a^4}{m}\left[d_1 + d_2 \tan\left(s_0 \ln\frac{|a|}{|a_0|} + \frac{\pi}{2}\right)\right], \quad (3)$$

where the numerical values of the universal constants d_1, $d_2 < 0$, and a_0 have been determined numerically recently by Braaten *et al.* [13]. Here a_0 is the value of the two-body scattering length for which the first three-body bound state with zero energy is formed. In the present parametrization a_0 replaces the logarithmic derivative and the matching radius used by Efimov [12,13]. Unlike d_1 and d_2, the parameter a_0 is system dependent and is also a genuine three-body characteristic. When approaching the three-body threshold (by changing the strength of the two-body interaction), just before the three-body bound state is formed, $g_3 \to -\infty$, in complete analogy with the behavior of a two-body scattering amplitude. After the three-body state has appeared g_3 is positive and $g_3 \to \infty$ when the threshold for the appearance of the three-body bound state is approached from the other side.

If the three-body zero-energy scattering amplitude g_3 is known, the contribution of the three-particle collisions to the ground state energy density of a dilute Bose gas can be evaluated in a similar manner as the leading order contribution of the two-body collisions

$$\mathcal{E}_3(\rho) = \frac{1}{6} g_3 \rho^3. \quad (4)$$

The experience gained from studying corrections to the main two-body contribution to the ground state energy of a dilute quantum gas [see Refs. [3–5] and Eq. (2)] shows that all other contributions are small in the dilute limit and they are not expected to lead to any qualitative changes in the properties of these systems. (In the case of fermion systems pairing correlations, however, could lead to a significant rearrangement of the ground state properties, but energetically the correction is typically small and the density profiles are not modified in any drastic manner; see, e.g., nuclei [14].) The $\mathcal{E}_3(\rho)$ contribution to the energy density can dominate the contribution arising from the two-body collisions if the argument of the tangent is close to $(2n + 1)\pi/2$ and the scattering length $a \approx a_0 \exp(n\pi/s_0)$, $n = 0, 1, \ldots$, is such that a three-body state is on the verge of appearing or it has just been formed. The situation is somewhat unique in this limit. With respect to the two-body collisions, the system is extremely dilute, but somewhat less dilute with respect to three-body collisions.

At the points where a three-body bound state appears, where g_3 becomes infinite and in the immediate neighborhood of them, the present calculational scheme fails, since the contribution of the three-body collisions has been evaluated only in the leading order of the gas approximation with respect to the three-body collisions. In the region near such poles, the contribution of the three-particle collisions could dominate over the genuine two-particle contributions, for appropriate values of ρ and a. In a trap the density profile of such a trapped Bose-Einstein condensed gas is given now by the new "Thomas-Fermi" formula

$$\rho(\boldsymbol{r}) = \sqrt{\frac{2[\mu - V_{\rm ext}(\boldsymbol{r})]}{g_3} + \left(\frac{g_2}{g_3}\right)^2} - \frac{g_2}{g_3}, \quad (5)$$

where $V_{\rm ext}(\boldsymbol{r})$ is the trapping potential and μ is the chemical potential. Besides obvious changes in the density profile of a trapped Bose-Einstein condensed gas, the spectrum of the elementary and collective excitations is, naturally, modified as well, as the compressibility of such a system is significantly affected by the three-body contribution to the ground state energy $\mathcal{E}_3$. I note here that the analyses of Refs. [15] consider a formally similar situation, but with a fictitious repulsive three-body force, whose nature and strength are never specified.

Since the two-body contribution to the ground state energy of a dilute Bose gas is negative, the three-body collisions in the regime where $g_3 > 0$ could lead to the stabilization of the system. What is particularly interesting for such a system is that a boson droplet—a boselet—could become self-bound and the trapping potential is not required anymore to keep the particles together. In the absence of the trapping potential and for a very large number of particles a boselet will have an almost constant density, corresponding to the infinite matter equilibrium density

$$\rho_0 = -\frac{3g_2}{2g_3}. \quad (6)$$

The ground state energy of an ensemble of N bosons in the mean-field approximation is given by [16]

$$E(N) = \int d^3r \varepsilon(\boldsymbol{r}) = \int d^3r \left[\frac{\hbar^2}{2m}|\nabla\psi(r)|^2 + \frac{1}{2}g_2\rho(\boldsymbol{r})^2 + \frac{1}{6}g_3\rho(\boldsymbol{r})^3\right], \quad (7)$$

where $\varepsilon(\boldsymbol{r})$ is the energy density and $\rho(\boldsymbol{r}) = |\psi(\boldsymbol{r})|^2$ is the number density. The density profile of semi-infinite matter can be shown to be given by

$$\rho(z) = |\psi(z)|^2 = \frac{\rho_0}{1 + \exp(2\kappa_0 z)}, \tag{8}$$

where

$$\kappa_0 = \sqrt{\frac{2m|\mu_0|}{\hbar^2}}, \qquad \mu_0 = -\frac{3g_2^2}{8g_3}, \tag{9}$$

z is the spatial coordinate normal to the surface and μ_0 is the chemical potential corresponding to infinite matter at equilibrium density ρ_0. One can now determine the surface tension σ of boselets from the obvious relation

$$\sigma = \int_{-\infty}^{\infty} dz[\varepsilon(z) - \mu_0\rho(z)] = \frac{g_3\rho_0^3}{12\kappa_0}. \tag{10}$$

The spectrum of both volume and surface sound waves can then be easily specified. The density profile of an infinite slab of finite width has a simple expression as well

$$\rho(z) = \rho_0 \frac{\mu}{\mu_0} \frac{1}{1 + \sqrt{1 - \frac{\mu}{\mu_0}}\cosh(2\kappa z)}, \tag{11}$$

where $\mu = -\hbar^2\kappa^2/2m$ and the chemical potential satisfies the restrictions $\mu_0 \le \mu < 0$. It is straightforward to show that for any slab with a width larger than its skin thickness, the quantities $(\mu - \mu_0)/\mu_0$ and $(\kappa_0 - \kappa)/\kappa_0$ are both exponentially small. These facts (along with numerical evidence not presented here) suggest that the basic properties of a N-particle boselet are given by the following relations:

$$\rho(\mathbf{r}) \approx \rho_0\left(1 + \frac{1}{2\kappa R}\right)\frac{1}{1 + \frac{\cosh(2\kappa r)}{\cosh(2\kappa R)}}, \tag{12}$$

$$R = r_0 N^{1/3} + r_1 N^{-1/3} + \mathcal{O}(N^{-2/3}), \tag{13}$$

$$E(N) = \mu_0 N + 4\pi r_0^2 \sigma N^{2/3} + \mathcal{O}(N^{1/3}), \tag{14}$$

$$\omega_l^2 = \frac{\sigma l(l-1)(l+2)}{m\rho_0 R^3} + \mathcal{O}(N^{-4/3}), \tag{15}$$

where $r_0 = (3/4\pi\rho_0)^{1/3}$, $\mu = dE(N)/dN = -\hbar^2\kappa^2/2m$, and ω_l is the frequency of the surface vibration mode with angular momentum l. The absence of the constant term in the expression for the radius was established numerically. The central density is larger then ρ_0 due to surface tension.

The possibility that the entire system can also undergo a transition to a gas phase of trimers cannot be ruled out at this time. Since there are no Efimov states for four or more particles, this trimer phase is perhaps unique. One cannot fail to see here an analogy with the Cooper pair–BEC crossover in the fermion case [17]. At each new three-body threshold, when g_3 becomes infinite, a new trimer phase appears, made of spatially larger trimers. The density drops naturally by a factor of 3 if a trimer phase is formed. The trimer-trimer scattering length is expected to be of the order of the trimer size, i.e., of order a, but the sign of this trimer-trimer scattering amplitude is so far unknown. Upon collapsing into trimers, the interaction energy decreases significantly, as now this energy is controlled by effective two-body processes only. If a trimer phase is formed, the size of the cloud in the trap should change abruptly.

It is obvious that in the case of a Fermi-Dirac system the role of three-body collisions could play an analogous role, if the Efimov effect could take place. Since three identical fermions could never be all in a relative s state, the particles should have either a spin larger than 1/2 or some other additional degree of freedom, like isospin in the case of nucleons. The characteristics of the Efimov effect for the case of particles with arbitrary spin and/or other discrete degrees of freedom (as well as arbitrary masses) have been established in Ref. [11]. Let me consider for illustrative purposes the case of two identical spin-1/2 fermion species π and ν. In the mean-field approximation the ground state energy can be evaluated using the following energy density functional

$$E(N_\pi, N_\nu) = \int d^3r\left\{\frac{\hbar^2}{2m}[\tau_\pi(\mathbf{r}) + \tau_\nu(\mathbf{r})] + \frac{g_2}{2}[\rho_\pi(\mathbf{r}) + \rho_\nu(\mathbf{r})]^2 - \frac{g_2}{4}[\rho_\pi(\mathbf{r})^2 + \rho_\nu(\mathbf{r})^2] + \frac{g_3}{4}\rho_\pi(\mathbf{r})\rho_\nu(\mathbf{r})[\rho_\pi(\mathbf{r}) + \rho_\nu(\mathbf{r})]\right\}, \tag{16}$$

where $\rho_{\pi,\nu}(\mathbf{r})$ are the corresponding number density distributions, and $N_{\pi,\nu} = \int d^3r\rho_{\pi,\nu}(\mathbf{r})$ and $\hbar^2\tau_{\pi,\nu}(\mathbf{r}) = \hbar^2\sum_{n=1}^{N_{\pi,\nu}/2} 2|\nabla\psi_n^{(\pi,\nu)}(\mathbf{r})|^2$ are the momentum distributions, defined through the corresponding single-particle wave functions $\psi_n^{(\pi,\nu)}(\mathbf{r})$.

Again, the most interesting regime to consider is that of a negative two-body scattering length $g_2 < 0$. If the three-body term is repulsive, that is, if $g_3 > 0$, a dilute Fermi-Dirac droplet will behave very much like a nuclear system. In this case the two-body effects will have an attractive role and the energy density functional of such a system will have the same qualitative structure as the popular Skyrme energy density functional in nuclear physics [18]. The Fermi-Dirac droplets—the fermilets—will have entirely unexpected properties: they will be self-bound and show saturation properties as well; see Fig. 1. The existence of an equilibrium state for infinite homogeneous fermionic matter is a sufficient condition for the existence of finite systems as well, see the case of boselets for example. Since one would be able to change the relative magnitude of the two-body and three-body interactions, the central density of a fermilet can be controlled. In the absence of Coulomb interaction the number of particles in a single fermilet is arbitrary. A particularly interesting aspect could be the interplay between the formation of Cooper pairs and

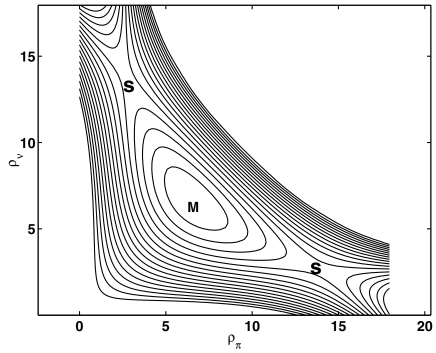

FIG. 1. A typical contour plot of the energy density for an homogeneous Fermi system consisting of two fermion species π and ν [see Eq. (16)] for the case $\hbar = m = g_3 = 1$ and $g_2 = -5$. Only the negative part of the energy density surface is plotted. The local minimum and the two saddle points are labeled by M and S, respectively. For other sets of parameters there is either only one saddle point or none at all.

fermionic trimers, since when $g_3 > 0$ a trimer bound state exists. The possibility to have a Fermionic system, with attractive two-body effective interactions and repulsive three-body effective interactions, thus opens the way towards the creation of "designer nuclei," an almost unthinkable flexibility, which could not be matched even by atomic clusters. Self-bound droplets of mixtures of fermions and bosons—ferbolets—are expected as well.

As a final remark, since recombination now requires four particles to collide, it is not unreasonable to expect rather long lifetimes for these new objects. Three-body recombination into deep two-body bound states (if such states exist) could, however, define the lifetime of these objects [6].

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