Rotating Spin-1 Bose Clusters

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We propose a simple scheme for generating rotating atomic clusters in an optical lattice which produces states with quantum Hall and spin liquid properties. As the rotation frequencies increase, the ground state of a rotating cluster of spin-1 Bose atoms undergoes a sequence of (spin and orbit) transitions, which terminates at an angular momentum L^* substantially lower than that of the boson Laughlin state. The spin-orbit correlations reflect ''fermionization'' of bosons facilitated by their spin degrees of freedom. We also show that the density of an expanding group of clusters has a scaling form which reveals the quantum Hall and spin structure of a single cluster.

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In this paper we would like to point out some remarkable features of rotating clusters of spin-1 Bose atoms which result from the coupling of quantum Hall physics to spin symmetry through Bose statistics. Our studies are prompted by the recent experiments on rotating Bose gases [1] and Bose-Hubbard systems [2], as well as the theoretical realization of a ''fragmented'' condensate in spin-1 Bose gas [3,4], and the theoretical demonstration that a three-dimensional Bose system can approach the quantum Hall limit at sufficiently large angular momenta [5]. As of now, one can release the spin degrees of freedom of a Bose gas in an optical trap [6], deposit a large amount of angular momentum into a Bose condensate }, and divide a Bose condensate into thousands of isolated clusters each with a few particles [2]. It is natural to ask whether these capabilities can be combined to produce a large collection of rotating and spin-carrying clusters. In fact, such possibilities are being considered in some laboratories [7].

There are many reasons for studying these Bose clusters. First, in these systems two types of strongly correlated ground states can be realized: quantum Hall droplets and fragmented Bose spin liquids. While the former is familiar to condensed matter physicists, the latter is new, with properties entirely different from the familiar single condensate (or coherent) states [3]. It is well known, however, that these states are very fragile as the number of particles increases, easily giving way to the highly robust coherent states [3,8]. On the other hand, such fragility disappears in small clusters, and the system becomes strongly correlated in this regime. Second, quantum clusters are amenable to exact treatments, which may reveal physical principles operative in a larger scale. Third, amid the growing interest in using bosons in lattices to process quantum information, the study of quantum clusters remains a basic step.

We present (i) a simple method to generate rotating clusters, (ii) the general features of their ground states, and (iii) the scaling behavior of the density of an expanding group of clusters, whose specific form reflects spin liquid and quantum Hall properties. Before proceeding, we first summarize the general features: (A) As the angular momentum L of the system increases, the interaction energy will be ''quenched'' to zero at a critical value L^*, considerably lower than that of the boson Laughlin state. (B) As L increases from 0 to L^*, the total spin of the ground state is specified by its orbital angular momentum, even though the interaction is invariant under separate spin and orbital rotations. (C) Both (A) and (B) reflect the phenomenon of ''fermionization'' facilitated by the internal degrees of the freedom (i.e., spin) of the system. This is achieved by populating bosons in different *spin-orbit* states, a process automatic for fermions. (D) The energetics leading to (A) and (B) also implies that, as the rotational frequency Ω increases toward the trap frequency ω, the angular momentum of the ground state will increase in discrete steps until it reaches L^*, and then remains constant until the system becomes unstable at $\Omega > \omega$. We shall now derive these results.

I. Rotating quantum clusters: their generation and Hamiltonian.—Consider particles in an optical lattice with an added rotating quadrupolar potential $W(\mathbf{r};t) = \lambda[r_1^2(t) - r_2^2(t)]$, where $r_i(t) = \mathbf{r}\cdot\hat{\mathbf{e}}_i(t)$, $\hat{\mathbf{e}}_1(t) = \hat{\mathbf{x}}\cos\Omega t + \hat{\mathbf{y}}\sin\Omega t$, $\hat{\mathbf{e}}_2(t) = \hat{\mathbf{y}}\cos\Omega t - \hat{\mathbf{x}}\sin\Omega t$, and λ is a constant. Such a potential can be generated by a pair of off-center rotating laser beams as in the Paris experiment [1]. The single-particle Hamiltonian is then $h(t) = \hat{T} + \hat{U} + W(\mathbf{r};t)$, where $\hat{T} = \mathbf{p}^2/2M$, and $\hat{U}$ is the periodic potential of the optical lattice. Deep in the Mott limit, $\hat{U}$ is an array of deep wells at lattice sites $\{\mathbf{R}\}$; the bottom of each is harmonic, $\hat{U}(\mathbf{r}) \to \hat{U}_{\rm har}(\mathbf{x}) = \frac{1}{2}M\omega^2\mathbf{x}^2$ as $\mathbf{r}\to\mathbf{R}$, $\mathbf{r} = \mathbf{x} + \mathbf{R}$. These wells are so deep that atoms in different wells are isolated from each other [9]. Shifting the origin to $\mathbf{R}$, the single-particle Schrödinger equation is $i\hbar\partial_t|\Psi_t\rangle = [\hat{T} + \hat{U}_{\rm har}(\mathbf{x}) + W(\mathbf{x}+\mathbf{R};t)]|\Psi_t\rangle$. Defining $|\Psi_t\rangle = e^{-i[\mathbf{a(t)}\cdot\mathbf{p}+\mathbf{b(t)}\cdot\mathbf{x}]/\hbar}e^{-i\theta(t)}e^{-i\Omega t\hat{L}_z}|\Phi_t\rangle$, and with appropriate choices of $\mathbf{a}$, $\mathbf{b}$, and θ [10], the Schrödinger equation can be written as $i\hbar\partial_t|\Phi_t\rangle = \overline{\mathcal{K}}|\Phi_t\rangle$, where $\overline{\mathcal{K}} = \hat{T} + \hat{U}_{\rm har}(\mathbf{x}) + W(\mathbf{x};0) - \Omega L_z$ is time independent, and $L_z = xp_y - yp_x$. Since $W(\mathbf{x},0)$ breaks cylindrical symmetry, it will impart angular momentum to the system. Once the system acquires angular momentum, cylindrical symmetry can be restored by turning off W, and $\overline{\mathcal{K}}$ reduces to the simple form $\mathcal{K} = \mathbf{p}^2/2M + \frac{1}{2}M\omega^2\mathbf{x}^2 - \Omega L_z$.

Including particle interactions, the Hamiltonian of a rotating spin-1 Bose cluster is, in first quantized form, $\hat{H} - \Omega\hat{L}_z = \hat{\mathcal{K}} + \hat{\mathcal{V}}$, where $\hat{\mathcal{K}} = \sum_i \hat{\mathcal{K}}_i$, $\hat{\mathcal{V}} = \sum_{i>j} \hat{\mathcal{V}}_{ij}$, $\hat{\mathcal{V}}_{ij} = (c_o + c_2\mathbf{F}_i \cdot \mathbf{F}_j)\delta(\mathbf{r}_i - \mathbf{r}_j)$, where c_0 and c_2 are density-density and spin-spin interactions [11,12]. Theoretical estimates show that $c_0 > 0$ for both ^{23}Na and ^{87}Rb; and that $c_2 > 0$ (antiferromagnetic) for ^{23}Na and $c_2 < 0$ (ferromagnetic) for ^{87}Rb [11]. Because of the similarity of scattering lengths in different angular momentum channels, it is found that $|c_2|/c_0 \sim 5\%$ for both cases. From now on, we shall focus on values of c_0 and c_2 appropriate for ^{23}Na and ^{87}Rb. For simplicity, we first consider zero magnetic field. Magnetic field effects will be considered at the end.

The eigenvalues of $\hat{\mathcal{K}}$ are [5] $E_{n,m;n_z}/\hbar = (\omega + \Omega)n + (\omega - \Omega)m + \omega n_z$, where n, m, and n_z are non-negative integers. For very tight traps, $\hbar\omega > Nc_0/a^3$ with $a = \sqrt{\hbar/M\omega}$, only states in the "lowest Landau level" (LLL), $(n = 0, m; n_z = 0)$, will appear in the ground states. The wave function of the state $(0, m; 0)$ is $w_m(x, y) f_0(z)$, where $w_m(x, y) = u^m e^{-|u|^2/2}/\sqrt{\pi a^2 m!}$, $f_0(z) = e^{-z^2/2a^2}/(\pi a^2)^{1/4}$, and $u = (x + iy)/a$. The many-body state is then

$$|\Phi\rangle = \overline{\int} B([u], [\mu]) \prod_{i=1}^{N} \hat{\psi}^{\dagger}_{\mu_i}(\mathbf{r}_i)|0\rangle, \qquad (1)$$

where $\overline{\int}(\cdots) \equiv \int \prod_{i=1}^{N} [d\mathbf{r}_i f_0(z_i) e^{-|u_i|^2/2}](\cdots)$, $[u] \equiv (u_1, u_2, \ldots u_N)$, $\mu_i = 1, 0, -1$ labels the spin of the ith boson, $[\mu] \equiv (\mu_1, \mu_2, \ldots, \mu_N)$, $B([u], [\mu])$ is a symmetric homogeneous polynomial of $[u]$ labeled by $[\mu]$, $\hat{\psi}(\mathbf{r})_\mu$ is the field operator in the LLL, $\hat{\psi}^{\dagger}_{\mu}(\mathbf{r}) = \sum_m w_m \times (x, y) f_0(z) a^{\dagger}_{m\mu}$, and $a^{\dagger}_{m\mu}$ is the creation operator for the state $(0, m; 0)$. In LLL, we have $\hat{\mathcal{K}} = \hbar(\omega - \Omega)\hat{L}_z$, $\hat{L}_z = \sum_{m=0}^{\infty} m a^{\dagger}_{m\mu} a_{m\mu}$. Since $[\hat{\mathcal{V}}, \hat{L}_z] = 0$, the ground state of the Hamiltonian $\hat{H} - \Omega\hat{L}_z$ can be determined once the spectrum of $\hat{\mathcal{V}}$ as a function of L_z ($\mathcal{V}_L$) is obtained.

Before proceeding, we define $A^x(\mathbf{r}) = [-\hat{\psi}_1(\mathbf{r}) + \hat{\psi}_{-1}(\mathbf{r})]/\sqrt{2}$, $A^y(\mathbf{r}) = -i[\hat{\psi}_1(\mathbf{r}) + \hat{\psi}_{-1}(\mathbf{r})]/\sqrt{2}$, and $A^z(\mathbf{r}) = \hat{\psi}_0(\mathbf{r})$. Under a spin rotation, $\vec{A}$ rotates as a 3D vector in (xyz) space. Thus, states with total spin ($S = 0$, 1, and 2) formed by two bosons at $\mathbf{r}_1$ and $\mathbf{r}_2$ are $\vec{A}^{\dagger}_1 \cdot \vec{A}^{\dagger}_2$, $\vec{A}^{\dagger}_1 \times \vec{A}^{\dagger}_2$, and $A^{\dagger}_{i1}A^{\dagger}_{j2} - \frac{1}{3}\delta_{ij}\vec{A}^{\dagger}_1 \cdot \vec{A}^{\dagger}_2$, where $\vec{A}_1 \equiv \vec{A}(\mathbf{r}_1)$.

II. The role of internal degrees of freedom and the "quenching" angular momentum L^*.—We have numerically diagonalized $\hat{\mathcal{V}}$ up to $N = 5$ particles. The spectrum $\mathcal{V}_L$ of the $N = 4$ cluster with $c_2 > 0$ is shown in Fig. 1(a). It shows a clear correlation between spin and orbital angular momentum in the ground state. In the following, we shall explain the origin, the systematics, and the analytic structures of these states.

Since $c_0 > |c_2|$, the minimum of the interaction $\hat{\mathcal{V}}_{ij}$ is zero, which occurs if the relative angular momentum ℓ between bosons i and j is nonzero. For scalar bosons, Bose statistics demands $\ell \geq 2$. A pair of spin-1 bosons, however, can make $\hat{\mathcal{V}}_{ij} = 0$ with $\ell = 1$ by making its spin part antisymmetric, i.e., forming a spin-1 state $\overline{\int}(u_1 - u_2)\vec{A}^{\dagger}_1 \times \vec{A}^{\dagger}_2|0\rangle$. This shows that angular momentum states forbidden to scalar bosons will become accessible in the presence of internal degrees of freedom. Relative angular momentum, however, costs kinetic energy. One will then have to balance potential and kinetic energy.

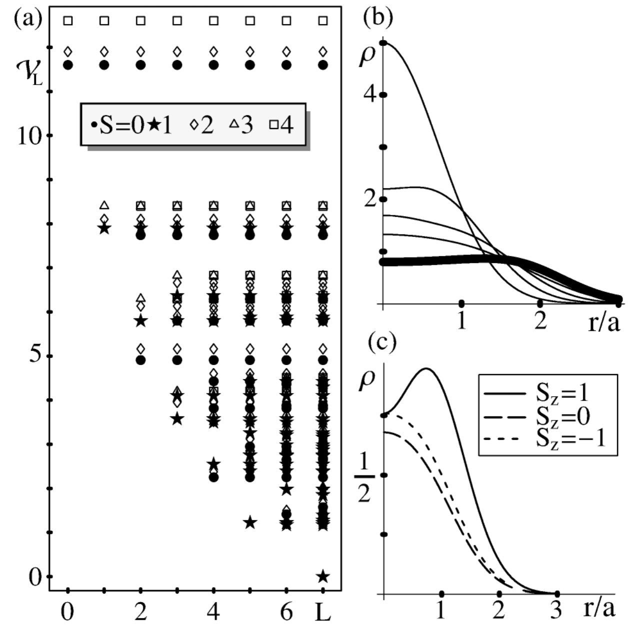

FIG. 1. $\mathcal{V}_L$ for an $N = 4$ cluster with $c_2 > 0$. (b) The column densities of a $N = 5$ cluster with with $L = 0, 3, 6, 9, 12$. The central density of these clusters decreases with increasing L. The $L = 12$ data is in bold. (c) The spin densities of the $L = 3$ cluster in (b); ($S = 1$). Densities, ρ, are scaled by $\rho_0 = 1/\pi a^2$.

Among all the ground states, the state with *minimum* angular momentum L^* that "quenches" interaction (i.e., making $\hat{\mathcal{V}} = 0$) plays a special role. It will be referred to as the "minimal quenching" state $|\Phi^*\rangle$. The vanishing of $\hat{\mathcal{V}}$ can occur only if no two bosons can occupy the same point in space, meaning that B in Eq. (1) contains a Laughlin factor $W[z] \equiv \Pi_{N \geq i > j \geq 1}(u_i - u_j)$. For spinless particles, Bose statistics demand that W appear twice; resulting in a minimal quenching state $B = W^2$, with $L^* = N(N - 1)$. For spin-1 bosons, the condition for quenching means

$$B([u], [\mu]) = W[u]F([u], [\mu]), \qquad (2)$$

where $F([u], [\mu])$ obeys Fermi statistics. Thus $L^* = N(N - 1)/2 + P$, where P is the lowest angular momentum possible for the fermionic wave function F.

Since there are only three distinct spin states for spin-1 bosons, the only clusters whose quenching "fermion" component F has $L = 0$ are $N = 2, 3$. For $N = 2$, we have $|\Phi^*\rangle^{(2)} = \overline{\int} u_{12}\vec{A}^{\dagger}_1 \times \vec{A}^{\dagger}_2|0\rangle$, where $u_{ij} \equiv u_i - u_j$. It has ($L^* = 1, S = 1$). For $N = 3$, we have $|\Phi^*\rangle^{(3)} = \overline{\int} u_{12}u_{23}u_{31}\vec{A}^{\dagger}_1 \times \vec{A}^{\dagger}_2 \cdot \vec{A}^{\dagger}_3|0\rangle$, with ($L^* = 3, S = 0$). For

$N > 3$, the quenching fermionic function F must have nonzero angular momentum; otherwise there will be two "fermions" occupying the same state. Particles must therefore populate different spin and orbital states. This procedure gives rise to a *unique* minimal quenching structure, whose general form will be clear after we enumerate a few cases. Writing $|\Phi^*\rangle^{(N)} = \overline{\int} W D^{(N)}|0\rangle$, we have

$$
\begin{aligned}
D^{(4)} &= [\vec{A}_1^\dagger \cdot \vec{A}_2^\dagger \times \vec{A}_3^\dagger](u_4\vec{A}_4^\dagger),\\
D^{(5)} &= [\vec{A}_1^\dagger \cdot \vec{A}_2^\dagger \times \vec{A}_3^\dagger](u_4\vec{A}_4^\dagger \times u_5\vec{A}_5^\dagger),\\
D^{(6)} &= [\vec{A}_1^\dagger \cdot \vec{A}_2^\dagger \times \vec{A}_3^\dagger](u_4\vec{A}_4^\dagger \times u_5\vec{A}_5^\dagger \cdot u_6\vec{A}_6^\dagger),\\
D^{(7)} &= [\vec{A}_1^\dagger \cdot \vec{A}_2^\dagger \times \vec{A}_3^\dagger][u_4\vec{A}_4^\dagger \times u_5\vec{A}_5^\dagger \cdot u_6\vec{A}_6^\dagger](u_7^2\vec{A}_7^\dagger),
\end{aligned}
\tag{3}
$$

which gives ($L^* = 7, S = 1$), ($L^* = 12, S = 1$), ($L^* = 18, S = 0$), and ($L^* = 26, S = 1$) for $N = 4, 5, 6$, and 7, respectively. In contrast, the corresponding L^* for scalar bosons are 12, 20, 30, 42, about 100% higher. Since W is antisymmetric, only the antisymmetric part of $D^{(N)}$will appear in $|\Phi^*\rangle^{(N)}$. The antisymmetrization turns the operator product into a determinant, just as fermion systems [13]. This prescription yields spin $S = 0, 1, 1$ for $N \equiv 0, 1, 2 (\mathrm{mod}\, 3)$ at quenching with $L^* = 2N^2/3 - N + S$.

Note that, as the particle number increases, more and more singlets made up of boson triplets appear in the minimal quenching state. This is a clear sign that the system becomes a spin liquid while evolving into a full-fledged quantum Hall state. Note also that the Laughlin factor W forces all ground states with $L > L^*$ to have zero interaction energy.

III. The route to quenching.—We now turn to the ground states with $L < L^*$. To illustrate the correlation between S and L, we denote the spin of the ground state of $\hat{\mathcal{V}}$ with angular momentum L as S_L, and display their value up to L^* for a cluster of N bosons as $[S_L]_N \equiv (S_0, S_1, ..., S_{L^*})_N$. For $c_2 > 0$, we have $(0, 1)_2$; $(1, 1, 1, 0)_3$, $(0, 1, 0, 1, 0, 1, 1, 1)_4$, $(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 1)_5$; and for $c_2 < 0$, we have $(2, 1)_2$, $(3, 2, 1, 0)_3$, $(4, 3, 2, 1, 2, 1, 1, 1)_4$, $(5, 4, 3, 3, 1, 1, 3, 1, 1, 1, 2, 2, 1)_5$.

To understand these structures, we note that, since $L < L^*$, there is not enough angular momentum to produce a Laughlin factor to prevent the coincidence of any two bosons. Still, it is desirable to generate as many boson pairs with *relative* angular momentum of unit strength as possible. This consideration then motivates some simple rules for possible candidates of the ground state and, for small clusters, they often narrow down to the exact structure. The rules are (i) for a given L, the ground state of interaction $\hat{\mathcal{V}}$ will contain a maximum number of pairs with unit *relative* angular momentum. (ii) For $c_2 > 0$ (or $c_2 < 0$), the ground state of $\hat{\mathcal{V}}$ will have a minimum (or maximum) total spin consistent with rule (i) and Bose statistics. We have constructed states according to these rules for clusters up to $N = 4$ bosons. The results are given in Table I, which reproduces the sequences $[S_L]^{(4)}$ listed above. When compared with the exact numerical results in the limit $|c_2|/c_0 \ll 1$, we found that most of states in Table I coincide with the exact numerical results, or otherwise have over 94% overlap with them. Since the interaction $\hat{\mathcal{V}}$ is invariant under separate spin and orbital rotation, the correlation between S and L comes entirely from statistics, just as in Hund's first and second rule.

In Fig. 1(b), we show the densities of a $N = 5$ cluster with $L = 0, 3, 6, 9, 12$, and in Fig. 1(c) we show the spin densities for the cluster with $N = 5, L = 3$, and $S = 1$. The emergence of a quantum Hall plateau in the density is clear near the quenching limit. (See also Section V.)

IV. The ground states as a function of Ω.—Experiments performed at fixed rotational frequency Ω are described by the Hamiltonian $\mathcal{H} \equiv H - \Omega L_z$. The ground state is given by the minimum of $\mathcal{H}_L = \mathcal{V}_L + \hbar(\omega - \Omega)L$, where $\mathcal{V}_L$ is the ground state energy of $\hat{\mathcal{V}}$ with $L_z = L$. If L were a continuous variable, and $\mathcal{V}_L$ a smooth curve, the optimum value L_0 will satisfy $\hbar(\omega - \Omega) = -\partial\mathcal{V}_L/\partial L$, and $\partial^2\mathcal{V}_L/\partial^2 L > 0$. In addition, we have $\partial\Omega/\partial L_0 = (\partial^2\mathcal{V}_L/\partial^2 L)_{L_o}$. From Fig. 1(a), we see that aside from a small number of cusps (in this case at $L = 5$ and 7) the envelop of $\mathcal{V}_L$ is smooth and concave up, implying that as Ω is increased the angular momentum

TABLE I. The ground states constructed from the rules in Section III. The ground state is written as $|\Phi\rangle = \overline{\int}\varphi_L D_S^{(N)\dagger}|0\rangle$, using $\Theta_{12} \equiv \vec{A}_1 \cdot \vec{A}_2$, $\Gamma_{123} \equiv \vec{A}_1 \times \vec{A}_2 \cdot \vec{A}_3$, $\vec{B}_{12} \equiv \vec{A}_1 \times \vec{A}_2$, $A_+ \equiv A_x + iA_y$, $w_{12...N} \equiv \prod_{N \geq i > j \geq 1}(u_i - u_j)$, $u_{ij} \equiv u_i - u_j$. All states listed are exact eigenstates except those with $L = 4, 5, 6$. For $L = 4$, the last factor of φ can be either u_{24} or u_{34}. The exact ground state is a combination of both. The state of $L = 5$ and 6 has 99.6% and 94% overlap with the exact ground state. The minimal quenching state is marked with an asterisk.

N	L	S	$D_S^{(N)}$, $c_2 > 0$	S	$D_S^{(N)}$, $c_2 > 0$	φ_L
2	0	0	Θ_{12}	2	$A_{1+}A_{2+}$	1
	1*	1	B_{12}	1	B_{12}	u_{12}
3	0	1	$\Theta_{12}\vec{A}_1$	3	$A_{1+}A_{2+}A_{3+}$	1
	1	1	$\vec{B}_{12} \times \vec{A}_3$	2	$B_{12+}A_{3+}$	u_{12}
	2	1	$\Theta_{12}\vec{A}_1$	2	$\Theta_{12}\vec{A}_1$	$u_{12}u_{13}$
	3*	0	Γ_{123}	0	Γ_{123}	w_{123}
4	0	0	$\Theta_{12}\Theta_{34}$	4	$\prod_{i=1}^4 A_{i+}$	1
	1	1	$\vec{B}_{12}\Theta_{34}$	3	$B_{12+}A_{3+}A_4$	u_{12}
	2	0	$\vec{B}_{12} \cdot \vec{B}_{34}$	2	$B_{12+}B_{34+}$	$u_{12}u_{34}$
	3	1	$\Gamma_{123}\vec{A}_4$	1	$\Gamma_{123}\vec{A}_4$	w_{123}
	4	0	$\vec{B}_{12} \cdot \vec{B}_{34}$	2	$B_{12+}B_{34+}$	$u_{12}u_{14}u_{23} \times (u_{24}, u_{34})$
	5	1	$\Gamma_{123}\vec{A}_4$	1	$\Gamma_{123}\vec{A}_4$	$w_{123}u_{24}u_{34}$
	6	1	$\Theta_{12}\vec{B}_{34}$	3	$A_{1+}A_{2+}B_{34+}$	$u_{12}^2u_{13}u_{14} \times u_{24}u_{34}$
	7*	1	$\Gamma_{123}(u_4\vec{A}_4)$	1	$\Gamma_{123}(u_4\vec{A}_4)$	w_{1234}

increases in small (but discrete) steps, reaching L^* at a critical frequency Ω^*.

For $\Omega > \Omega^*$ the minimum of $\mathcal{H}_L$ within the interval $(0 < L < L^*)$ still occurs at L^*. However, over the interval $L > L^*$, we have $\mathcal{V}_L = 0$, and hence $\mathcal{H}_L = \hbar(\omega - \Omega)L$ has a minimum at L^*. This establishes statement (D) that, for sufficiently rapid rotation, the angular momentum saturates at L^*.

V. The spin and quantum Hall signature of rotating clusters.—Because of the large number (Q) of identical but phase incoherent clusters in the insulating phase, many quantities measured for the entire group of clusters coincide with the *quantum mechanical* averages of the same quantity within a single cluster, except that it is magnified by Q. This is best illustrated by considering the density of the entire group of clusters after the trap is turned off. Because of tight trapping, the expansion is mainly driven by the localization energy of the trap and is therefore ballistic. The field operator for the cluster (say, at $\mathbf{R} = 0$) then evolves as $\hat{\psi}_\mu(\mathbf{r}) = \int d\mathbf{r}' G(\mathbf{r} - \mathbf{r}'; t) \times \hat{\psi}_\mu(\mathbf{r}')$, where $G(\mathbf{r}; t) = \prod_{i=1}^3 \tilde{G}(r_i; t)$, $\tilde{G}(x; t) = (M/2\pi i\hbar t)^{1/2} \exp[iMx^2/2\hbar t]$. Before the expansion, the field operator is $\hat{\psi}_\mu(\mathbf{r}) = \sum_\ell w_\ell(\mathbf{r}_\perp) f_0(z) a_{\ell\mu}$, and the single-particle density matrix of the cluster is

$$\rho_{\mu\nu}(\mathbf{r}) \equiv \langle \hat{\psi}_\nu^\dagger(\mathbf{r}) \hat{\psi}_\mu(\mathbf{r}) \rangle = g_{\mu\nu}(\mathbf{r}_\perp) v(z), \qquad (4)$$

$$g_{\mu\nu}(\mathbf{r}_\perp) = \sum_\ell |w_\ell(\mathbf{r}_\perp)|^2 \langle a_{\ell\nu}^\dagger a_{\ell\mu} \rangle, \qquad (5)$$

with $v(z) = |f_0(z)|^2$, and $\mathbf{r}_\perp = (x, y)$. In the insulating regime, the density matrix of the entire cluster collection is $\rho_{\mu\nu}^{\rm all}(\mathbf{r}) = \sum_{\mathbf{R}} \rho_{\mu\nu}(\mathbf{r} - \mathbf{R})$, where the sum is over all clusters. After the expansion, we have $\hat{\psi}_\mu(\mathbf{r}, t) = \sum_\ell \overline{w}_\ell(\mathbf{r}_\perp) \overline{f}_0(z) a_{\ell\mu}$, $\overline{w}_\ell(\mathbf{r}_\perp) = \int \tilde{G}(x - x'; t) \tilde{G}(y - y'; t) w_\ell(\mathbf{r}'_\perp)\, d\mathbf{r}'_\perp$, and $\overline{f}_0(z) = \int \tilde{G}(z - z'; t) f_0(z')\, d\mathbf{x}'$. This implies $\rho_{\mu\nu}(\mathbf{r}, t) \equiv \langle \hat{\psi}_\nu^\dagger(\mathbf{r}t) \hat{\psi}_\mu(\mathbf{r}t) \rangle = \overline{g}_{\mu\nu}(\mathbf{r}_\perp) \overline{v}(z)$, where $\overline{g}_{\mu\nu}(\mathbf{r}_\perp)$ and $\overline{v}(z)$ are $g_{\mu\nu}(\mathbf{r}_\perp)$ and $v(z)$ in Eq. (4) with w_ℓ and f_0 replaced by $\overline{w}_\ell$ and $\overline{f}_0$. By evaluating $\overline{g}_{\mu\nu}$ and $\overline{v}$, we find that $\overline{g}_{\mu\nu}(\mathbf{r}) = \lambda(t)^{-2} g_{\mu\nu}[\mathbf{r}/\lambda(t)]$, $\overline{v}(z) = \lambda(t)^{-1} v[z/\lambda(t)]$, where $\lambda(t) = \sqrt{1 + (\omega t)^2}$. We then have

$$\rho_{\mu\nu}^{\rm all}(\mathbf{r}, t) = \lambda(t)^{-3} \sum_{\mathbf{R}} \rho_{\mu\nu}\{[\mathbf{r} - \mathbf{R}]/\lambda(t)\}. \qquad (6)$$

The diagonal elements of Eq. (6) give the density of each spin component. Equation (6) shows that the density after expansion is simply a sum of all the expanded clusters, each of which obeys a scaling relationship. For times $t \gg 1/\omega$, each cluster has expanded to a size much larger than the dimension (Δ) of the origin cluster ensemble; Eq. (6) is then well approximated by $\rho_\mu^{\rm all}(\mathbf{r}, t) = Q/(\omega t)^3 \rho_\mu(\mathbf{r}/\omega t)$ up to a correction of order $\mathcal{O}(\Delta/a\omega t)$. The density of each spin component of the entire expanded ensemble at long times therefore reproduces that of each individual cluster. (For magnetic field effects, see [14].)

The identity $\sum_\mu \int \hat{\psi}_\mu^\dagger(\mathbf{r})\, (r/a)^2 \hat{\psi}_\mu(\mathbf{r}) = \hat{L}_z + \hat{N}$ in the LLL implies that, if L is the angular momentum of a cluster before expansion, then the mean square radius at time $t \gg 1/\omega$ is $\overline{r^2} = \int r_\perp^2 \rho^{\rm all}(\mathbf{r}, t) = Qa^2(\omega t)^2(L + N)$, and $\overline{z^2} = \int z_\perp^2 \rho^{\rm all}(\mathbf{r}, t) = Qa^2(\omega t)^2 N/2$. One can therefore extract the angular momentum of the system from the shape of the entire expanded clusters. Moreover, the difference of the mean squared radius between successive L states as Ω increases is $Qa_\perp^2(\omega t)^2$.

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[10] $\dot{\mathbf{a}} = -\mathbf{b}/M$, $\dot{\mathbf{b}} = M\omega^2\mathbf{a} + 2\lambda[(R_1 + a_1)\hat{\mathbf{e}}_1 - (R_2 + a_2)\hat{\mathbf{e}}_2]$, $\hbar\dot{\theta} = \lambda[R_1^2 - R_2^2 - a_1^2 + a_2^2]$, $R_i = \mathbf{R} \cdot \mathbf{e}_i$, $a_i = \mathbf{a} \cdot \mathbf{e}_i$.
[11] T. L. Ho, Phys. Rev. Lett. **81**, 742 (1998).
[12] T. Ohmi and K. Machida, J. Phys. Soc. Jpn. **67**, 1822 (1998).
[13] For example, the antisymmetrization of $D^{(7)}$, denoted as $\tilde{D}^{(7)}$, $\tilde{D}^{(7)} = \sum_P (-1)^P A_{P1}^{x\dagger} A_{P2}^{y\dagger} A_{P3}^{z\dagger} (u_{P4} A_{P4}^{x\dagger})(u_{P5} A_{P5}^{y\dagger}) \times (u_{P6} A_{P6}^{z\dagger})(u_{P7}^2 A_{P7}^{x\dagger})$, where P is permutation of the particles. The generalization of this construction to arbitrary cluster is obvious.
[14] If the expansion takes place in the presence of a magnetic field $\mathbf{B}$, and if the original trap has cylindrical symmetric along z, $V = \frac{1}{2}M(\omega_\perp^2 r_\perp^2 + \omega_z^2 z^2)]$, Eq. (6) will be generalized to $\rho_\mu^{\rm all}(\mathbf{r}, t) = \frac{Q}{\lambda_z \lambda_\perp^2} \hat{U}_{\mu\alpha} [\sum_{\mathbf{R}} \rho_{\alpha\beta}(\frac{\mathbf{r}_\perp - \mathbf{R}_\perp}{\lambda_\perp}, \frac{z - R_z}{\lambda_z})] \times \hat{U}_{\beta\mu}^\dagger$, $\lambda_{\perp,z}(t) = \sqrt{1 + (\omega_{\perp,z} t)^2}$, $\hat{U} = e^{-i\vec{\theta} \cdot \mathbf{F}}$, $\vec{\theta} = \gamma \mathbf{B} t$, where γ is the gyromagnetic ratio. At long times ($\omega_\perp t \gg 1$, $\omega_z t \gg 1$), we have $\rho_\mu^{\rm all}(\mathbf{r}, t) = [Q/(\omega_z \omega_\perp^2 t^3)] \hat{U}_{\mu\alpha} \times \rho_{\alpha\beta}(\mathbf{r}_\perp/\omega_\perp t, z/\omega_z t) \hat{U}_{\beta\mu}^\dagger$, which provides information about the density matrix of the preexpansion cluster.
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