Quantum Optical Communication Rates through an Amplifying Random Medium

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We study the competing effects of stimulated and spontaneous emission on the information capacity of an amplifying disordered waveguide. At the laser threshold the capacity reaches a "universal" limit, independent of the degree of disorder. Whether or not this limit is larger or smaller than the capacity without amplification depends on the disorder, as well as on the input power. Explicit expressions are obtained for heterodyne detection of coherent states, and generalized for an arbitrary detection scheme.

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To faithfully transmit information through a communication channel, the rate of transmission should be less than the capacity of the channel [1,2]. Although current technology is still far from the quantum limit, there is an active scientific interest in the fundamental limitations to the capacity imposed by quantum mechanics [3,4]. Ultimately, these limitations originate from the uncertainty principle, which is the source of noise that remains when all external sources have been eliminated.

An important line of investigation deals with strategies to increase the capacity. One remarkable finding of recent years has been the beneficial role of multiple scattering by disorder, which under some circumstances can increase the capacity by increasing the number of modes that effectively carry the information [5,6]. Quite generally, the capacity increases with increasing signal-to-noise ratio, so that amplification of the signal is a practical way to increase the capacity. When considering the quantum limits, however, one should include not only the amplification of the signal (e.g., by stimulated emission), but also the excess noise (e.g., due to spontaneous emission). The two are linked at a fundamental level by the fluctuation-dissipation theorem, which constrains the beneficial effect of amplification on the capacity [7].

While the effects of disorder and amplification on communication rates have been considered separately in the past, their combined effects are still an open problem. Even the basic question, "Does the capacity go up or down with increasing gain?", has not been answered. We were motivated to look into this problem by the recent interest in so-called "random lasers" [8,9]. These are optical media with gain, in which the feedback is provided by disorder instead of by mirrors. Below the laser threshold, these materials behave similar to linear amplifiers with strong intermode scattering, and this results in some unusual noise properties [10,11]. As we will show here, the techniques developed in connection with random lasers can be used to predict under what circumstances the capacity is increased by amplification.

We consider the transmission of information through a linear amplifier consisting of an N-mode waveguide that is pumped uniformly over a length L (see Fig. 1). We will refer to amplification by stimulated emission, but one can equally well assume other gain mechanisms (for example, stimulated Raman scattering [12]). The amplification occurs at a rate $1/\tau_a$. The waveguide also contains passive scatterers, with a transport mean-free path l. The combined effects of scattering and amplification are described by a $2N \times 2N$ scattering matrix S which is superunitary ($SS^\dagger - \mathbb{1}$ positive definite).

We assume that the information itself is of a classical nature (without entanglement of subsequent inputs), but fully account for the quantum nature of the electromagnetic field that carries the information. The quantized radiation is described by a vector $a^{\rm in}$ of bosonic annihilation operators for the incoming modes and a vector $a^{\rm out}$ for the outgoing modes. The two vectors are related by the input-output relation [10,13,14]

$$a^{\rm out} = Sa^{\rm in} + Ub^\dagger. \tag{1}$$

The vector of bosonic creation operators $b^\dagger$ describes spontaneous emission by the amplifying medium. The fluctuation-dissipation theorem relates U to S by

$$UU^\dagger = SS^\dagger - \mathbb{1}. \tag{2}$$

The first communication channel that we examine is heterodyne detection of coherent states [3]. The sender uses a single narrow-band mode α (with frequency ω_0 and bandwidth $\Delta\omega$), to transmit a complex number μ by means of a coherent state $|\mu\rangle$ (such that $a^{\rm in}_\alpha|\mu\rangle = \mu|\mu\rangle$). The receiver measures a complex number ν by means of heterodyne detection of mode β. Two sources of noise may cause ν to differ from μ: spontaneous emission by the amplifying medium, and nonorthogonality of the two coherent states $|\mu\rangle$ and $|\nu\rangle$, described by the overlap

$$|\langle\mu\,|\,\nu\rangle|^2 = \pi^{-1}\exp(-|\mu - \nu|^2). \tag{3}$$

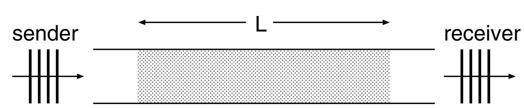

FIG. 1. Communication channel consisting of an N-mode waveguide that is amplifying over a length L. Both sender and receiver use a single narrow-band mode (indicated by a plane wave).

The *a priori* probability $p(\mu)$ that the sender transmits the number μ, and the conditional probability $\mathcal{P}(\nu\,|\,\mu)$ that the receiver detects ν if μ is transmitted, determine the mutual information [3],

$$I = \int d^2\nu \int d^2\mu\, \mathcal{P}(\nu\,|\,\mu)p(\mu)\log_2\left(\frac{\mathcal{P}(\nu\,|\,\mu)}{\tilde{p}(\nu)}\right). \quad (4)$$

We have defined $\tilde{p}(\nu) = \int d^2\mu\, \mathcal{P}(\nu\,|\,\mu)p(\mu)$. The channel capacity C (in bits per use) is obtained by maximizing I over the *a priori* distribution $p(\mu)$, under the constraint of fixed input power $P = P_0 \int d^2\mu |\mu|^2 p(\mu)$ (with $P_0 = \hbar\omega_0\Delta\omega/2\pi$). As argued in Ref. [15], any randomness in the scattering medium that is known to the receiver but not to the sender can be incorporated by averaging I before maximizing; hence,

$$C = \max_{p(\mu)}\langle I\rangle. \quad (5)$$

The brackets $\langle\cdots\rangle$ indicate an average over different positions of the scatterers.

The calculation of the capacity is greatly simplified by the fact that the spontaneous emission noise is a Gaussian superposition of coherent states. This is expressed by the density matrix of the amplifying medium,

$$\rho_{\text{medium}} \propto \int d^2\vec{\beta}\, \exp(-|\vec{\beta}|^2/f)\,|\vec{\beta}\rangle\langle\vec{\beta}|, \quad (6)$$

where $\vec{\beta}$ is a vector of $2N$ complex numbers and $|\vec{\beta}\rangle$ is the corresponding coherent state (such that $b_n|\vec{\beta}\rangle = \beta_n|\vec{\beta}\rangle$). The variance $f = N_{\text{upper}}(N_{\text{upper}} - N_{\text{lower}})^{-1}$ depends on the degree of population inversion of the upper and lower atomic levels that generate the stimulated emission. Minimal noise requires a complete population inversion: $N_{\text{lower}} = 0 \Rightarrow f = 1$. We consider that case.

We similarly assume that heterodyne detection adds the minimal amount of noise to the signal. (This requires that the image band is in the vacuum state [3].) The conditional probability is then given by a projection,

$$\mathcal{P}(\nu\,|\,\mu) = \langle\nu|\rho_{\text{out}}(\mu)\,|\nu\rangle, \quad (7)$$

of the density matrix $\rho_{\text{out}}(\mu)$ of the outgoing mode β onto the coherent state $|\nu\rangle$ (for an incoming coherent state $|\mu\rangle$ in mode α). In view of Eqs. (1) and (6), we have

$$\rho_{\text{out}}(\mu) \propto \int d^2\nu' \exp\left(-\frac{|\nu' - S_{\beta\alpha}\mu|^2}{\sum_n |U_{\beta n}|^2}\right)|\nu'\rangle\langle\nu'|. \quad (8)$$

This is again a Gaussian superposition of coherent states, but now the variance is related by Eq. (2) to the scattering matrix of the medium: $\sum_n |U_{\beta n}|^2 = \sum_n |S_{\beta n}|^2 - 1$.

Substituting ρ_{out} into Eq. (7), and using Eq. (3), we arrive at

$$\mathcal{P}(\nu\,|\,\mu) \propto \exp\left(-\frac{|\nu - S_{\beta\alpha}\mu|^2}{\sum_n |S_{\beta n}|^2}\right). \quad (9)$$

This expression for the conditional probability has the same Gaussian form as in previous studies [15,16] of communication channels degraded by Gaussian noise, but the essential difference is that in our case the noise strength is not independent of the transmitted power, but related to it by the fluctuation-dissipation theorem (2).

The calculation of the capacity proceeds as in Refs. [15,16]. The optimum *a priori* distribution $p(\mu) \propto \exp(-|\mu|^2 P_0/P)$ is independent of the scattering matrix S, so the maximization and disorder average in Eq. (5) may be interchanged. The result is

$$C = \langle\log_2(1 + \mathcal{R})\rangle, \qquad \mathcal{R} = \frac{(P/P_0)\,|S_{\beta\alpha}|^2}{\sum_{n=1}^{2N} |S_n|^2}. \quad (10)$$

The quantity $\mathcal{R}$ is the signal-to-noise ratio at the receiver's end. We can write $\mathcal{R}$ equivalently in terms of the transmission matrix t (from sender to receiver) and the reflection matrix r (from receiver to receiver):

$$\mathcal{R} = \frac{(P/P_0)\,|t_{\beta\alpha}|^2}{\sum_{n=1}^{N}(|t_{\beta n}|^2 + |r_{\beta n}|^2)}. \quad (11)$$

In the absence of intermode scattering, one has $|t_{nm}|^2 = \delta_{nm}$ and $r_{nm} = 0$; hence, $\mathcal{R} = \delta_{\alpha\beta}P/P_0$ and $C = \log_2(1 + \delta_{\alpha\beta}P/P_0)$, independent of the amount of amplification. The increase in capacity by stimulated emission is canceled by the extra noise from spontaneous emission [7]. In the absence of amplification, but in the presence of scattering, one has $\sum_n |S_{\beta n}|^2 = 1$; hence, $C = \langle\log_2(1 + |t_{\beta\alpha}|^2 P/P_0)\rangle$. The capacity is reduced by intermode scattering in the same way as for the lossy channel studied in Ref. [17].

The average over the scatterers can be done analytically in the limit $N \gg 1$ of a large number of modes in the waveguide. Sample-to-sample fluctuations in the denominator $\sigma = \sum_n(|t_{\beta n}|^2 + |r_{\beta n}|^2)$ are smaller than the average by an order N, so these fluctuations may be neglected and we can replace the denominator by its average $\bar{\sigma}$. The fluctuations in the numerator $\tau = |t_{\beta\alpha}|^2$ cannot be ignored. These are described (for $N \gg 1$) by the Rayleigh distribution $\mathcal{P}(\tau) = \bar{\tau}^{-1}e^{-\tau/\bar{\tau}}$. Integrating $\log_2[1 + (P/P_0)\tau/\bar{\sigma}]$ over τ with distribution $\mathcal{P}(\tau)$, we arrive at

$$C = e^{\mathcal{R}_{\text{eff}}^{-1}}\Gamma(0; \mathcal{R}_{\text{eff}}^{-1})/\ln 2, \qquad \mathcal{R}_{\text{eff}} = \frac{P\bar{\tau}}{P_0\bar{\sigma}}, \quad (12)$$

with $\Gamma(0;x)$ the incomplete gamma function. The dependence of the capacity C on the effective signal-to-noise ratio $\mathcal{R}_{\text{eff}}$ is plotted in Fig. 2. It lies always below the capacity $C_0 = \log_2(1 + \mathcal{R}_{\text{eff}})$, which one would obtain by ignoring fluctuations in τ. For $\mathcal{R}_{\text{eff}} \ll 1$ the two capacities approach each other, $C \approx C_0 \approx \mathcal{R}_{\text{eff}}/\ln 2$, while for $\mathcal{R}_{\text{eff}} \gg 1$ one has $C_0 \approx \log_2\mathcal{R}_{\text{eff}}$ versus $C \approx \log_2\mathcal{R}_{\text{eff}} - \gamma/\ln 2$ (with $\gamma \approx 0.58$ Euler's constant).

The quantity $\mathcal{R}_{\text{eff}}$ depends on three length scales [11]: the length L of the amplifying region, the mean-free path l, and the amplification length $l_a = \sqrt{D\tau_a}$ (with D the diffusion constant). The two averages $\bar{\tau}$, $\bar{\sigma}$ can be calculated from the diffusion equation in the regime $l \ll l_a, L$. There is a weak dependence on the mode indices α, β in

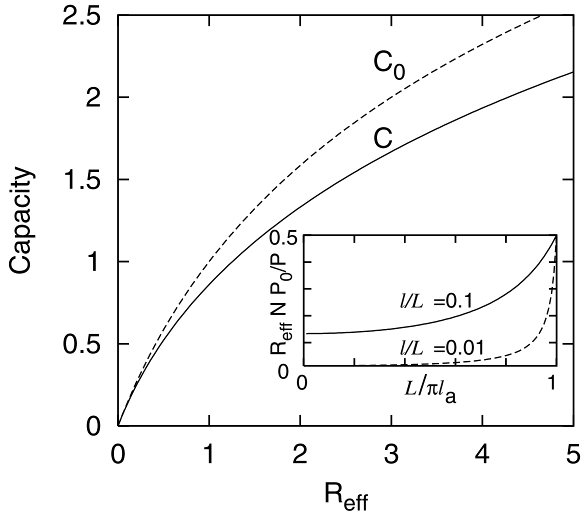

FIG. 2. Capacity C for heterodyne detection of coherent states as a function of the signal-to-noise ratio $\mathcal{R}_{\rm eff}$. The result (12) lies below the value $C_0 = \log_2(1 + \mathcal{R}_{\rm eff})$ that ignores statistical fluctuations. Inset: Dependence of $\mathcal{R}_{\rm eff}$ on the relevant length scales.

this diffusive regime, which we ignore. The result is

$$\bar{\tau} = \frac{4l/3l_a}{N\sin(L/l_a)}, \qquad \bar{\sigma} = 1 + (4l/3l_a)\frac{1-\cos(L/l_a)}{\sin(L/l_a)}. \tag{13}$$

The effective signal-to-noise ratio,

$$\mathcal{R}_{\rm eff} = \frac{P}{NP_0}[1 - \cos(L/l_a) + (3l_a/4l)\sin(L/l_a)]^{-1}, \tag{14}$$

is plotted in Fig. 2 (inset). Without amplification, for $l_a \gg L$, one has $\mathcal{R}_{\rm eff} = \frac{4}{3}(l/NL)P/P_0$. Amplification increases $\mathcal{R}_{\rm eff}$, up to the limit $\mathcal{R}_{\rm eff} \to P/2NP_0$ that is reached upon approaching the laser threshold $l_a \to L/\pi$.

We conclude that amplification in a disordered waveguide increases the capacity for heterodyne detection of coherent states, up to the limit

$$C_\infty = e^{2NP_0/P}\Gamma(0; 2NP_0/P)/\ln 2, \tag{15}$$

at the laser threshold. This limit is "universal," in the sense that it is independent of the degree of disorder (as long as we remain in the diffusive regime). We have $C_\infty \approx P/2NP_0 \ln 2$ for $P \ll NP_0$ and $C_\infty \approx \log_2(P/2NP_0) - \gamma/\ln 2$ for $P \gg NP_0$. The increase in the capacity by amplification in the diffusive regime is therefore up to a factor $3L/8l$ for $P \ll NP_0$ and up to a factor $1 + (\ln L/l)(\ln P/NP_0)^{-1}$ for $P \gg NP_0(L/l)$. All this is in contrast to the case of a waveguide without disorder, where the capacity is independent of the amplification.

We now relax the requirement of heterodyne detection and instead consider the maximum communication rate for any physically possible detection scheme [3]. We still assume that the information is encoded in coherent states, and use the same Gaussian *a priori* distribution $p(\mu) \propto \exp(-|\mu|^2 P_0/P)$ as before. It has been conjectured [18] that an input of coherent states with this Gaussian distribution actually maximizes the information rate for any method of nonentangled input with a fixed mean power (the so-called one-shot unassisted classical capacity).

The capacity for an arbitrary detection scheme is given by the Holevo formula [19,20],

$$C_{\rm H} = H\left(\int d^2\mu\, p(\mu)\rho_{\rm out}(\mu)\right) - \int d^2\mu\, p(\mu) H[\rho_{\rm out}(\mu)],$$

where $H(\rho) = -{\rm Tr}\rho\log_2\rho$ is the von Neumann entropy. For a Gaussian density matrix $\rho \propto \int d^2\mu \exp(-|\mu - \mu_0|^2/x)$, one has [21]

$$H(\rho) = (x+1)\log_2(x+1) - x\log_2 x \equiv g(x). \tag{16}$$

Applying this formula to the Gaussian $\rho_{\rm out}(\mu)$ in Eq. (8) and the Gaussian $p(\mu)$, we arrive at the capacity

$$C_{\rm H} = g(\tau P/P_0 + \sigma - 1) - g(\sigma - 1). \tag{17}$$

For a channel without amplification $\sigma \to 1$ and so $C_{\rm H} = g(\tau P/P_0)$, which lies above the capacity for heterodyne detection considered earlier. At the other extreme, upon approaching the laser threshold, $\sigma \to \infty$ and we have $C_{\rm H} \to \log_2(\tau P/\sigma P_0)$, which is the same limiting expression as for heterodyne detection.

The average over disorder can be carried out as previously by replacing σ by $\bar{\sigma}$ and averaging over τ with the Rayleigh distribution $\mathcal{P}(\tau)$. The result is

$$C_{\rm H} = \frac{\bar{\tau}P}{P_0}\log_2\frac{\bar{\sigma}}{\bar{\sigma}-1} + \frac{\bar{\tau}P}{P_0\ln 2} \times [e^{\mathcal{R}_{\rm eff}^{-1}}\Gamma(0;\mathcal{R}_{\rm eff}^{-1}) - e^{\mathcal{R}_{\rm eff}'^{-1}}\Gamma(0;\mathcal{R}_{\rm eff}'^{-1})], \tag{18}$$

where $\mathcal{R}_{\rm eff}/\mathcal{R}'_{\rm eff} = 1 - 1/\bar{\sigma}$.

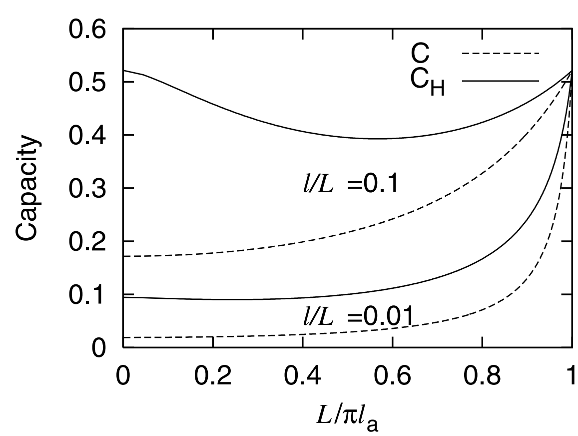

FIG. 3. Amplification dependence of the capacity C for heterodyne detection of coherent states [Eq. (12)] and the capacity $C_{\rm H}$ for arbitrary detection [Eq. (18)]. The input power is fixed at $P/P_0N = 1$ and two values of l/L are chosen.

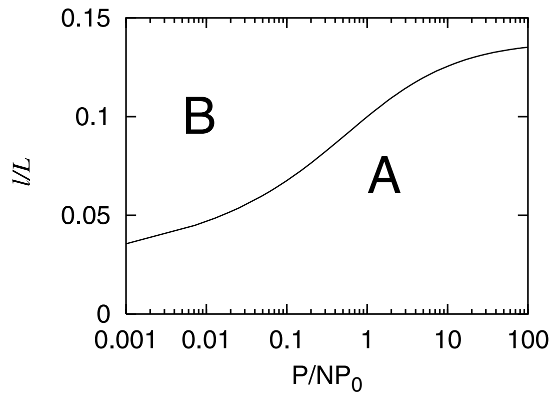

FIG. 4. Curve in parameter space separating region A [in which $C_\infty > C_{\rm H}(0)$] from region B [in which $C_\infty < C_{\rm H}(0)$]. In region A amplification of sufficient strength increases the capacity $C_{\rm H}$, while in region B it does not.

As shown in Fig. 3, the dependence of $C_{\rm H}$ on the amount of amplification is nonmonotonic—in contrast to the monotonically increasing C. Weak amplification reduces the capacity $C_{\rm H}$, while stronger amplification causes $C_{\rm H}$ to rise to the limit C_∞ at the laser threshold. The initial decrease for $l_a \gg L$ is described by

$$C_{\rm H}(L/l_a) \approx C_{\rm H}(0) - (4lL^2/3l_a^2)\log_2(\pi l_a/L). \quad (19)$$

Whether or not amplification ultimately increases $C_{\rm H}$ depends on the degree of disorder and on the input power. We indicate by A the region in parameter space where $C_\infty > C_{\rm H}(0)$ and by B the region where $C_\infty < C_{\rm H}(0)$. In region A strong amplification increases $C_{\rm H}$ while in region B it does not. The separatrix is plotted in Fig. 4. For $P/NP_0 \ll 1$, the analytical expression for this curve separating regions A and B is $P/NP_0 = (3L/4l)\exp(-3L/8l + \gamma)$, while for $P/NP_0 \gg 1$ we find a saturation at $l/L = 3/8e \approx 0.14$. This means that for $P/NP_0 \gg 1$ strong amplification increases the capacity $C_{\rm H}$ provided $l < 0.14L$.

At the laser threshold, both C and $C_{\rm H}$ reach the same universal limit C_∞ given by Eq. (15), which depends only on the dimensionless input power per mode P/NP_0 and not on the degree of disorder. This remarkably rich interplay of multiple scattering and amplification is worth investigating experimentally, for example, in the context of a random laser [8,9].

In conclusion, we have investigated the effect of amplification on the information capacity of a disordered waveguide, focusing on the competing effects of stimulated and spontaneous emission. We have compared the capacity C for heterodyne detection of coherent states with the Holevo bound $C_{\rm H}$ for an arbitrary detection scheme. While amplification increases C for any magnitude of disorder and input power, the effect on $C_{\rm H}$ can be either favorable or not, as is illustrated by the "phase diagram" in Fig. 4.

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