

Uniqueness of the Entanglement Measure for Bipartite Pure States and Thermodynamics

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(Received 4 January 2002; published 27 June 2002)

We apply the axiomatic approach to thermodynamics presented by Giles to derive a unique measure of entanglement for bipartite pure states. This implies that local manipulations of entanglement in quantum information theory and adiabatic transformations of states in thermodynamics have the same underlying mathematical structure. We discuss possible extensions of our results to mixed and multipartite states.

DOI: 10.1103/PhysRevLett.89.037903

PACS numbers: 03.65.Ud, 03.67.-a, 05.70.-a

An understanding of entanglement and its characterization form the cornerstone of the new and rapidly growing field of quantum information and computation [1]. We need to know how much entanglement is at our disposal since entanglement is a form of resource that can enhance information processing [2]. Although a great deal of work has recently been performed in this direction [3], it is widely acknowledged that we do not have a complete understanding of even the bipartite entanglement for mixed states (i.e., states consisting of two subsystems usually considered to be shared between two parties). There are a number of measures to quantify entanglement which apply in different settings and have different properties [3]. The consensus, however, is that local operations aided with classical communication (LOCC) are the key to explaining entanglement [4–6]. They are able to separate disentangled states from entangled states and thus introduce a *directionality* to entanglement manipulation processes: an entangled state can always be converted to a disentangled one by LOCC, but not vice versa.

A comparison with thermodynamics will be very helpful at this point. The second law of thermodynamics tells us which (energy conserving) processes are allowed in nature *without any reference to the underlying physical structure*. The central role is played by adiabatic processes and entropy is used to separate the possible from the impossible processes according to a very simple principle: if a state A has more entropy than B , then there is an adiabatic process to go from B to A , but not vice versa. This was first clearly realized by Caratheodory [7], who restated the second law by saying that in the neighborhood of any state there exist states which are adiabatically inaccessible from it. This allowed him to derive an entropy function which is able to introduce ordering into the set of physical states. The second law thus tells us that adiabatic processes cannot decrease entropy of the system itself. The question, then, is whether entropy is the only such function. To answer this question, however, thermodynamics needed first to be put onto a more secure mathematical foundation. In the words of Caratheodory himself: “What Thermodynamics needs is the establishment of logical order, essentially an intellectual cleanup. This is a problem for a mathematician. The

fundamental ideas and concepts have been introduced by physicists long ago and a mathematician need not worry about it.” The first such formalization came from Giles in 1964 [8] and was recently extended by Lieb and Yngvason [9].

An ideal physical theory should consist of two independent parts: a mathematical theory and a set of rules of interpretation of various mathematical objects involved in the theory. By formalizing a physical theory in such a way as to divorce it completely from the physical interpretation, it is possible to derive a mathematical structure that may be useful in a completely different physical setting to the original one. The question we address in this Letter is whether it is possible to apply the thermodynamical reasoning of Giles and Lieb and Yngvason to entanglement manipulations to derive a unique measure of the amount of entanglement in a general setting. The analogy between thermodynamics and quantum information would identify adiabatic processes with LOCCs and entropy with entanglement. The aim is to order quantum states according to whether they can be converted into each other by LOCCs. We use Giles’s approach to show that the same abstract structure of thermodynamics captures entanglement manipulations in quantum mechanics. As a consequence we can derive a rigorous proof for the unique measure of entanglement for pure states in the same way that a unique measure of order (entropy) is constructed in thermodynamics. We note that the approach of Giles is more suitable for entanglement manipulations than the approach of Lieb and Yngvason. This is because, as we see later, Giles bases his approach on the transformation of a discrete number of copies of a physical state, while Lieb and Yngvason rephrase the axioms in terms of continuous mixtures of states. The latter could also be applied to entanglement, but it is in our view not as naturally suited as Giles’s treatment.

Our abstract approach to entanglement is different from the existing method where one looks for a minimal number of conditions for a measure of entanglement that would single out a unique one [3,6,10,11]. The existing method has a strong flavor of Shannon’s pioneering approach to information theory [12]. Shannon considered functions on the set of probability distribution which would describe

their information content. By introducing three natural conditions that this function should satisfy he arrived at a unique measure, called the Shannon entropy. These conditions are remarkably similar to the conditions leading to a unique measure of entanglement for pure bipartite states [10]. This, of course, is not surprising. It is well known that the Shannon entropy of the probabilities derived from Schmidt coefficients in the Schmidt decomposition [13] of a pure bipartite state is a good measure of entanglement [4].

The remaining part of the Letter is structured as follows. First, we introduce the formal theory of Giles and we list the set of axioms he introduced to capture thermodynamical processes. Second, we define the (generalized) notion of LOCC which allows us to apply the presented axioms to entanglement manipulations. Finally, we compare our approach to the other existing approach and discuss the domain of its applicability.

To begin, we study a nonempty set S , whose elements are called *states*, in which two operations, $+$ and \rightarrow , are defined. States are denoted by a, b, c, \dots . A *process* is an ordered pair of states (a, b) . In the following axioms we omit the phrase “for all . . .”

Axioms 1–5: (1) *The operation $+$ is associative and commutative*; (2) $a \rightarrow a$; (3) $a \rightarrow b$ and $b \rightarrow c \implies a \rightarrow c$; (4) $a \rightarrow b \iff a + c \rightarrow b + c$; (5) $a \rightarrow b$ and $a \rightarrow c \implies b \rightarrow c$ or $c \rightarrow b$. Let us briefly discuss why these axioms describe the structure of thermodynamics. The operation $+$ represents the physical operation of considering two systems together. Therefore it must naturally be associative and commutative. The operation \rightarrow represents an adiabatic process which is meant to convert different physical states into each other. Therefore, like any other physical process, it should naturally be reflective and transitive as in axioms 2 and 3. Axiom 4 is the first nonintuitive property linking the operations $+$ and \rightarrow . In the forward direction it is obvious that if state a can be converted into b then the presence of another state c should not alter this fact; i.e., we can convert a and c into b and c by converting a into b while doing nothing to c . In the backward direction, however, this axiom is not completely obvious. It says that if a process is possible with the aid of another state, then we, in fact, do not need this other state for the process. Thermodynamics deals with macroscopic systems with a large number of degrees of freedom (subsystems). It is in the “asymptotic” limit that this axiom becomes more natural. Finally, axiom 5 is the key property which allows us to compare different states and processes. It says that any two states that are accessible from a third state must be accessible to each other at least in one direction. Not being able to do so would lead to states which would be incomparable as there would be no physical way of connecting them. Thus a unique way of ordering states would be impossible.

We now need the following definition in order to introduce the remaining two axioms. It is important for

comparing the sizes (contents/amounts) of different physical states.

Definition: Given states a and b we write $a \subset b$ (and say that a is contained in b), if there exists a positive integer n and a state c such that

$$na + c \rightarrow nb \quad \text{or} \quad nb \rightarrow na + c.$$

This really says that the state a is smaller than b if a requires the help of another state c to be converted to or derived from b .

Definition: A state e is an *internal* state if, given any state x , there exists a positive integer n such that $x \subset ne$.

This definition serves to introduce a reference state, which is the one that can contain any other physical state given sufficiently many copies of it. The concept of the internal state is necessary to give a basic metric unit (yardstick) to quantify the physical content of a state in a unique way (i.e., independent of states).

Now we state the remaining two axioms.

Axioms 6–7: (6) *There exists an internal state*; (7) *given a process (a, b) , if there exists a state c such that for any positive real number ϵ there exists positive integers m, n and states x, y such that $m/n < \epsilon$, $x \subset mc$, $y \subset mc$, and $na + x \rightarrow nb + y$, then $a \rightarrow b$* . Axiom 6 is necessary if we are to compare contents of different states in a unique way. Axiom 7 is the most complex axiom in the theory, although it is strongly motivated by the logic of thermodynamical reasoning. Loosely speaking, it states that if we can transform a into b with an arbitrarily small environmental influence, then this influence can be ignored. This, in some sense, introduces continuity into thermodynamical properties (and will be crucial for entanglement manipulations later). A reader interested in a more detailed physical interpretation of the axioms is advised to consult Giles’s book [8].

The above seven axioms are the crux of Giles’s formal theory which capture the key idea behind the second law of thermodynamics. Based on them Giles proves the existence of the unique *entropy function*. However, his formalization is general enough to be applicable to local manipulations of quantum states leading to a rigorous proof of the existence of the unique entanglement function for pure states.

Now to apply the axioms to entanglement manipulations consider S to be the set of all bipartite quantum pure states and the operation $+$ to be the tensor product \otimes (note that \otimes is commutative providing that when we exchange quantum states we also exchange the operations on them). In order to define the operation \rightarrow , we first give the definition of \subset in the quantum setting and then extend the concept of LOCCs in the spirit of axiom 7. In the standard (nonextended) definition, LOCC consists of the most general operations that the two parties can perform locally on their subsystems aided with classical communication [6].

Definition: We say that a pure state a is contained in a pure state b , designated as $a \subset b$, if and only if (iff) there

exists an integer n and a state c such that for every small real ϵ we can find an LOCC map, Φ , for which we have either

$$\begin{aligned} & \|\Phi(a^{\otimes n} \otimes c) - b^{\otimes n}\| < \epsilon \\ \text{or } & \|\Phi(b^{\otimes n}) - a^{\otimes n} \otimes c\| < \epsilon. \end{aligned}$$

Thus we say that a quantum state a is contained within a state b if, with the help of some other state c , a can be transformed by LOCC into b . Now we define what we mean by a transformation of one quantum state into another. Our definition is motivated by Giles's suggestion [8] to regard axiom 7 as a more general definition of the arrow which captures real physical processes more precisely: we usually transform a large number of copies of a system into a large number of copies of another system and this happens with the aid of an environment.

Definition: We say that a pure state a can be converted into a pure state b , designated as $a \rightarrow b$, iff for every small real ϵ , there exists a state c , such that for every small real δ there exist integers n, m and an LOCC map, Φ , and states x and y satisfying the following property:

$$\begin{aligned} m/n < \delta, \quad x \subset mc, \quad y \subset mc \\ \text{and } & \|\Phi(a^{\otimes n} \otimes x) - b^{\otimes n} \otimes y\| < \epsilon. \end{aligned}$$

We can see two important points in applying Giles's formalism to entanglement manipulations. One is that the approach is naturally asymptotic. This is of no surprise if we expect a unique measure of entanglement to emerge. In fact, if we deal with a finite number of copies [14], then we know that axiom 4 does not hold in the backward direction. Namely, we can have a situation where a cannot be transformed into b , but this could be achieved with the presence of an additional state c (known as the catalyzer [15]). Point two is that our definition of LOCC is not only asymptotic but allows the presence of an additional system as long as the state returned at the end of the process is included in the same state as the original [16]. Therefore our definition of LOCC is tailored so that it naturally satisfies all the axioms of Giles. What remains to be shown is that these axioms lead to a unique measure of entanglement imposed on pure quantum states. Following Giles's approach we first need to define the notion of the "component of content function." This notion is important as the resulting entanglement measure will be unique up to the addition of this function. The uniqueness we refer to is up to an affine transformation. In simple terms this means that the "origin" level of entropy is not fixed (i.e., we can choose any number to represent the entropy of the internal state) and also it can be scaled by an arbitrary real number (i.e., the entropy function can be multiplied and this will not affect its ordering properties).

Definition: A real-valued function Q defined for every state is a *component of content function* if (i) $Q(a \otimes b) =$

$Q(a) + Q(b)$; (ii) $a \rightarrow b \implies Q(a) = Q(b)$. By entanglement measure we mean the following:

Definition: A real-valued function E defined for every state is a *entanglement function* if (i) $E(a \otimes b) = E(a) + E(b)$; (ii) if $a \rightarrow b$ and $b \rightarrow a$, then $E(a) = E(b)$; (iii) if $a \rightarrow b$ and $b \not\rightarrow a$, then $E(a) > E(b)$. The following two theorems, stated here without proof, characterize the entanglement function (see Giles [8], theorems A4.2 and A.4.11).

Theorem 1: Let S_1 be an entanglement function. If Q is a component of content and λ a positive real number, then $\lambda S_1 + Q$ is an entanglement function; moreover, any entanglement function S may be written in this form.

Theorem 2: There exists a positive entanglement function.

The proof is constructive and can be found in [8]. It leads to a definition for a measure of entanglement, which in context of theorem 1 is unique. Giles's definition (of entropy) when applied to entanglement measures with the internal state being a maximally entangled state becomes simpler than his original definition involving an arbitrary internal state. The basic reason for this is that our definition of arrow satisfies a stronger condition than stated in axiom 5. Namely, given any two pure bipartite states b and c , we have that either $b \rightarrow c$ or $c \rightarrow b$. The definition of the amount of entanglement of a now becomes

$$E(a) = \inf\{m/n: y \subset e^{\otimes m}, y \rightarrow a^{\otimes n} \otimes x\},$$

where the infimum is taken over all integers m, n and all states x, y and e is any internal state. This definition of entanglement has a strong resemblance to the notions of entanglement of formation [17]. Indeed, suppose that state e is some maximally entangled state of two qubits. Then the above measure of entanglement of state a looks at the (asymptotic) minimal number of copies, m , of state e we need to invest to obtain n copies of a , by LOCC with the aid of a catalyzer. Note that we may well define the amount of entanglement of a state a to be the maximum number m of maximally entangled states we can obtain from n copies of a when n is large. In this case we have the entanglement of distillation (interestingly, this is a concept that Giles does not consider in thermodynamics). Otherwise, we can use any other internal state e (e.g., a nonmaximally entangled state) and all the resulting measures of entanglement would be equivalent up to an affine transformation.

We now compare our approach with the other existing approach for the uniqueness of the entanglement measure for pure states due to Donald *et al.* [11]. Building on the results of Popescu and Rohrlich [5] and Vidal [18], they show that there are three conditions necessary for the existence of a unique entanglement measure for pure states: (i) additivity, (ii) monotonicity, and (iii) asymptotic continuity. The unique measure resulting from these three axioms is equal to the von Neumann entropy of entanglement. Note that if in our definition of entanglement e is taken to be a maximally entangled state, then the amount of

entanglement becomes $E(a) = \inf\{m/n: e^{\otimes m} \rightarrow a^{\otimes n} \otimes x\}$. It is very simple to show that this quantity is equal to the von Neumann entropy of entanglement. This therefore offers a different way of proving that Giles's axioms lead to a unique measure of entanglement.

In Giles's formalism there is no reference to the number of subsystems or their dimensionality involved in entanglement manipulations. Thus, it may be tempting to conclude that there is a unique measure of entanglement for pure states of multiparty systems. However, it is in general not clear that axiom 5 will hold for more than two subsystems. For example, suppose that state b is a tripartite state $|\psi_1\rangle \otimes (|0_2 0_3\rangle + |1_2 1_3\rangle)$, while state c is $(|0_1 0_2\rangle + |1_1 1_2\rangle) \otimes |\psi_3\rangle$. Then these states cannot be converted into each other by our definition of \rightarrow (where each party 1, 2, 3 acts locally with the help of catalyzers and is allowed to classically communicate to other parties) even though both can be derived from a common state a of the form $|0_1 0_2 0_3\rangle + |1_1 1_2 1_3\rangle$. This suggests that either the notion of a unique measure of entanglement is misplaced for more than two parties or that we have to extend the notion of \rightarrow to satisfy axiom 4. The latter is, however, difficult to imagine as we are already allowing quite a wide range of local operations.

The situation is somewhat similar in the bipartite mixed state case. Axiom 5 may not be satisfied in general although there is no proof of this. If axiom 5 fails, this would imply that there are states b and c which cannot be compared by converting them into each other even though they can be derived from a common source. This in turn means that arbitrary order may be assigned to them, i.e., according to one measure b would be more entangled than c and according to some other measure it could be the other way around without contradicting other axioms. Virmani and Plenio [19] have already shown that if two measures of entanglement coincide on pure states and order mixed states differently, then they must be different measures (i.e., they cannot be equivalent up to an affine transformation). Therefore we may again be in a situation that if we want to have a unique measure of entanglement we need to extend the definition of \rightarrow . A possible avenue to explore is to consider positive partial transposition processes defined by Rains [20].

In this Letter we have shown that the set of axioms originally presented by Giles to capture the second law of thermodynamics can be applied to entanglement manipulations. This leads to a unique measure of entanglement for pure states in the same way that Giles showed the existence of a unique measure of order in thermodynamics. In order to satisfy the axioms we had to define the notion

of local operations to include the possibility of catalyzers. Once catalyzers and arbitrarily large numbers of copies are allowed in manipulations of entanglement we see that Giles's axioms are naturally satisfied. This then establishes the fact that not only are there analogies between thermodynamics and entanglement [5,6,21], but that the two, in fact, have *exactly* the same underlying structure. Therefore, it is of no surprise that same entropic measure is used to order both thermodynamical states as well as quantum entangled states. Our work suggests that this approach may be extended to consider mixed and multiparty quantum states.

We thank J. Eisert, M. Plenio, and G. Vidal for useful discussions on the subject of this Letter. This work is funded by EPSRC and by European grant EQUIP (Contract No. IST-1999-11053).

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