Activating Distillation with an Infinitesimal Amount of Bound Entanglement

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We show that bipartite quantum states of any dimension, which do not have a positive partial transpose (NPPT), become 1-distillable when one adds an infinitesimal amount of bound entanglement. To this end we investigate the activation properties of a new class of symmetric bound entangled states of full rank. It is shown that in this set there exist universal activator states capable of activating the distillation of any NPPT state. The result shows that even a small amount of bound entanglement can be useful for quantum information purposes.

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Introduction.—The possibility of *distillation* plays a crucial role in quantum communication and quantum information processing (cf. [1]). Together with quantum error correction it enables all the fascinating applications provided by quantum information theory in the presence of a noisy and interacting environment. Despite its practical relevance and quite considerable effort in that direction, however, many of the basic questions concerning distillation are yet unanswered. Most notable is the question whether a given quantum state is distillable or not, i.e., whether it is possible to obtain pure maximally entangled states from several copies of it by means of local operations and classical communication (LOCC).

A necessary condition for the distillability of a state described by a density matrix ρ is the fact that its partial transpose ρ^{T_A}, defined with respect to a given product basis by $\langle ij|\rho^{T_A}|kl\rangle = \langle kj|\rho|il\rangle$, has a negative eigenvalue [2]. Except for special cases like states on $\mathbb{C}^2 \otimes \mathbb{C}^n$ [3,4] and Gaussian states [5], it is, however, unclear whether this condition is sufficient as well. There is some evidence presented in [4,6] that this may not be the case and that there are indeed undistillable states, whose partial transpose is not positive (NPPT). At least there exist n-undistillable NPPT states for every finite n, meaning that no LOCC operation on n copies leads even to a single entangled two qubit state [4,6].

However, if we enlarge the class of allowed distillation protocols from LOCC to channels respecting the positivity of the partial transpose, then every NPPT state becomes 1-distillable [7,8] (which can be shown by using *entanglement witnesses* [8]). Moreover, it is a result from [9] that these channels can always be stochastically implemented by an LOCC operation where the two parties are given an entangled state with positive partial transpose (PPT) as an additional resource. The latter is known to be *bound entangled* since the entanglement needed for the preparation of the state cannot be recovered by distillation [2]. Nevertheless, PPT bound entangled states can be useful in order to *activate* the distillability of bipartite NPPT states [10,12].

The aim of the present paper is to investigate the limits and requirements of such an activation process. We will show that there exist states with an arbitrary small amount of PPT bound entanglement, which are capable of activating any NPPT state. The required additional resource is therefore universal as well as arbitrarily weakly entangled.

Preliminaries on symmetric states.—One of the key ideas in what follows will be the exploitation of the symmetry properties of states commuting with certain local unitaries. Two well known one-parameter families of such states are the *Werner states* and *isotropic states*, both playing an important role in the sequel.

Werner states [16] acting on a Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ with dimensions $\dim\mathcal{H}_A = \dim\mathcal{H}_B = d$ commute with all unitaries of the form $U \otimes U$ and can be written as

$$\rho(\alpha) = \left(\mathbf{1} - \frac{\alpha}{d}\mathbb{F}\right)/(d^2 - \alpha), \qquad \alpha \in [-d, d], \tag{1}$$

with $\mathbb{F}$ being the *flip operator* defined with respect to some product basis by $\mathbb{F}|ij\rangle = |ji\rangle$. A Werner state is entangled iff $\alpha \in (1, d]$ and 1-distillable iff $\alpha \in (d/2, d]$. Moreover, it was shown in [14] that any NPPT state can be mapped onto an entangled Werner state by means of LOCC operations. Therefore we can in the following restrict our discussion to the activation of Werner states keeping in mind that the obtained results hold for any NPPT state.

Isotropic states [14,15] commuting with all unitaries of the form $U \otimes \bar{U}$ (where $\bar{U}$ is the complex conjugate of U) are combinations of the maximally mixed state $\mathbf{1}/d^2$ and the projector $\mathbb{P} = |\Omega\rangle\langle\Omega|$ onto the maximally entangled state $|\Omega\rangle = 1/\sqrt{d}\sum_{i=1}^{d}|ii\rangle$:

$$\omega(f) = f\mathbb{P} + \frac{1-f}{d^2-1}(\mathbf{1} - \mathbb{P}), \qquad f \in [0, 1]. \tag{2}$$

An isotropic state ω is known to be 1-distillable iff the *maximally entangled fraction* $f = \langle\Omega|\omega|\Omega\rangle > 1/d$, which is a sufficient condition for any other state as well [14]. Hence, an activation protocol succeeded if this condition is fulfilled by the final state.

The symmetric states playing the central role in the present paper act on a larger Hilbert space $\mathcal{H} = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2}$ of total dimension d^4, where A and B again label the two parts of the system situated at different locations. The symmetry group under consideration is the group of all unitaries of the form $W = (U \otimes V)_A \otimes (U \otimes \bar{V})_B$. States σ commuting with all these unitaries can most easily be expressed in terms of the minimal projectors $\{P_i\}$ spanning the commutant of the group [16]:

$$\forall\, W : [\sigma, W] = 0 \Leftrightarrow \sigma = \sum_{i=1}^{4} \lambda_i P_i / \mathrm{tr}[P_i], \tag{3}$$

with

$$P_2^1 = \frac{1}{2}(1 \mp \mathbb{F})_1 \otimes \mathbb{P}_2\,, \tag{4}$$

$$P_4^3 = \frac{1}{2}(1 \mp \mathbb{F})_1 \otimes (1 - \mathbb{P}_2)\,. \tag{5}$$

Note that, as labeled by the indices, the tensor products correspond to a split $1|2$ (and not $A|B$). Positivity and normalization of σ requires $\lambda_i \geq 0$ and $\sum_{i=1}^4 \lambda_i = 1$ such that any state of the considered symmetry can be characterized by a vector $\vec{\lambda} \in \mathbb{R}^3$ lying in a tetrahedron, which is given by these constraints. We note further that the set of symmetric states in (3) is Abelian; i.e., all symmetric states commute with each other.

The *activation protocol* we use follows closely an idea of Ref. [9]. Initially the two parties A and B are supposed to share a Werner state $\rho(\alpha)$ acting on $\mathcal{H}_0 = \mathcal{H}_{A_0} \otimes \mathcal{H}_{B_0}$ with $\dim\mathcal{H}_0 = d^2$ and a symmetric state σ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ given by Eq. (3). After a local filtering operation is applied by projecting onto maximally entangled states $\mathbb{P}_{A_{0,1}}$ and $\mathbb{P}_{B_{0,1}}$ (acting on $\mathcal{H}_{A_0} \otimes \mathcal{H}_{A_1}$ and $\mathcal{H}_{B_0} \otimes \mathcal{H}_{B_1}$, respectively) the maximally entangled fraction of the resulting state on system 2 is given by

$$f(\rho(\alpha);\sigma) := \frac{\mathrm{tr}[(\rho(\alpha) \otimes \sigma)(\mathbb{P}_{A_{0,1}} \otimes \mathbb{P}_{B_{0,1}} \otimes \mathbb{P}_2)]}{\mathrm{tr}[(\rho(\alpha) \otimes \sigma)(\mathbb{P}_{A_{0,1}} \otimes \mathbb{P}_{B_{0,1}} \otimes \mathbf{1}_2)]}\,. \tag{6}$$

We know that σ *activates* $\rho(\alpha)$ if $f(\rho(\alpha);\sigma) > 1/d$. Since the output state of the protocol is itself isotropic, this condition is also necessary for the activation.

Of course, we are interested only in cases where $\alpha \in (1, d/2]$, i.e., $\rho(\alpha)$ is entangled but not 1-distillable, and σ is in turn a PPT bound entangled state. The latter requires the classification of the symmetric states in (3), which is the content of the next section.

Classification and Activation.—The following discussion will mainly take place in the three dimensional space given by the expansion coefficients $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ from Eq. (3). A vector $\vec{\lambda}$ corresponds to a (positive and normalized) symmetric state $\sigma(\vec{\lambda})$ iff $\vec{\lambda} \in S = \{\vec{v} \in \mathbb{R}^3 | v_i \geq 0, \sum_i v_i \leq 1\}$, where the state space S is a tetrahedron.

The set $\mathcal{P}$ corresponding to normalized operators with a positive partial transpose can easily be obtained by observing that the symmetry group of the partially transposed operators σ^{T_A} in (3) is equal to the group of unitaries W when interchanging the systems $1 \leftrightarrow 2$. The respective minimal projectors $\{Q_i\}$ can therefore be obtained from the projectors $\{P_i\}$ simply by relabeling the systems $1 \leftrightarrow 2$, and the kth coordinate of an extreme point $\vec{p}^{(i)}$ of $\mathcal{P}$ is thus given by

$$p_k^{(i)} = \mathrm{tr}[Q_i^{T_A} P_k]/\mathrm{tr}[Q_i]\,. \tag{7}$$

Hence, $\mathcal{P}$ is again a tetrahedron and Eq. (7) leads to the extreme points:

$$\vec{p}^{(1)} = \left(\frac{d-1}{2d}, \frac{-d-1}{2d}, \frac{1-d^2}{2d}\right),$$

$$\vec{p}^{(2)} = \left(\frac{1-d}{2d}, \frac{1+d}{2d}, \frac{-(d-1)^2}{2d}\right),$$

$$\vec{p}^{(3)} = \left(\frac{-1}{2d}, \frac{-1}{2d}, \frac{d+1}{2d}\right),$$

$$\vec{p}^{(4)} = \left(\frac{1}{2d}, \frac{1}{2d}, \frac{d-1}{2d}\right).$$

Straightforward linear algebra now allows us to compute the extreme points $\{\vec{\tau}^{(i)}\}$ of the intersection $S \cap \mathcal{P}$ corresponding to the set of symmetric PPT states, which is shown in Fig. 1:

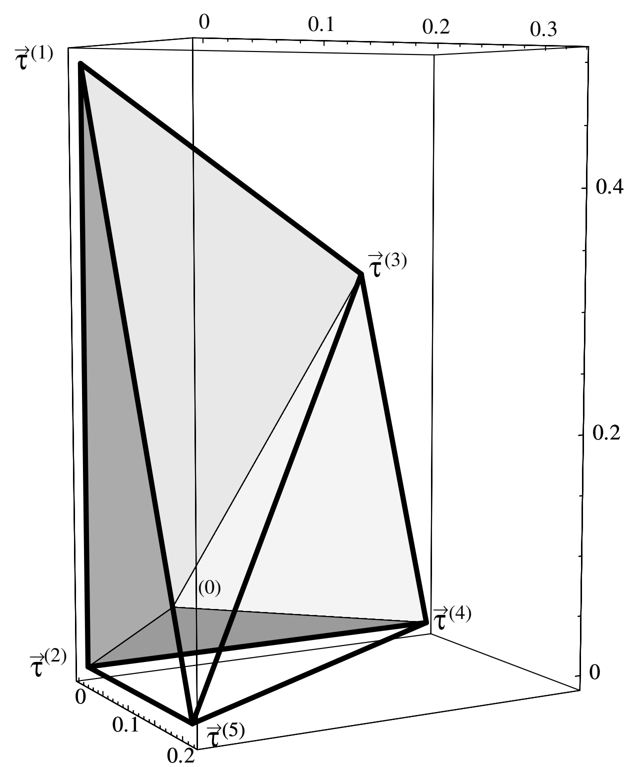

FIG. 1. The set of symmetric PPT states σ (thick wired object) parametrized by the three coordinates $\lambda_i = \mathrm{tr}[\sigma P_i]$, plotted for $d = 3$. The solid object inside corresponds to the set of separable states. The *universal activators* lie on the plane $\{\vec{\tau}^{(3)}, \vec{\tau}^{(4)}, \vec{\tau}^{(5)}\}$ and contain an arbitrary small amount of entanglement near the line $\{\vec{\tau}^{(3)}, \vec{\tau}^{(4)}\}$.

$$\vec{\tau}^{(1)} = \left(0, 0, \frac{1}{2}\right), \qquad \vec{\tau}^{(2)} = (0, 0, 0),$$

$$\vec{\tau}^{(3)} = \left(\frac{1}{2d}, \frac{1}{2d}, \frac{d-1}{2d}\right), \qquad \vec{\tau}^{(4)} = \left(0, \frac{1}{d}, 0\right),$$

$$\vec{\tau}^{(5)} = \left(\frac{1}{d+2}, 0, 0\right).$$

The PPT state space given by the convex hull of these points can be divided into three parts: separable states, as well as activating and not activating bound entangled states.

(i) Separable states: A vector $\vec{\lambda} \in \mathcal{S} \cap \mathcal{P}$ corresponds to a separable symmetric state iff we can find any (not necessarily symmetric) separable state ρ_{sep} such that $\lambda_i = \text{tr}[P_i \rho_{\text{sep}}]$ (cf. [15]).

A special case of a symmetric separable state is, of course, a tensor product of a Werner and an isotropic state $\sigma = \rho(\alpha)_1 \otimes \omega(f)_2$, where both are separable. In fact, if we choose these two states lying on the separable boundaries, i.e., $\alpha \in \{-d, 1\}$ and $f \in \{0, 1/d\}$, we retrieve the extreme points $\vec{\tau}^{(1)}, \ldots, \vec{\tau}^{(4)}$.

Another point $\vec{\tau}^{(0)} = (\frac{1}{2d}, 0, \frac{d-2}{4d})$ that will also turn out to be an extreme point of the set of separable symmetric states is obtained from the product state:

$$\rho_{\text{sep}} = |\Phi^+\rangle\langle\Phi^+|_A \otimes |\Psi^-\rangle\langle\Psi^-|_B, \tag{8}$$

where $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |22\rangle)$ and $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|12\rangle - |21\rangle)$ are two-dimensional maximally entangled states.

The convex hull of the points $\vec{\tau}^{(0)}, \ldots, \vec{\tau}^{(4)}$ (see Fig. 1) already covers the entire separable region as we are going to show in the following that the complement of this polytope within $\mathcal{S} \cap \mathcal{P}$ corresponds to bound entangled states.

(ii) Bound entangled and activating: The equation $f(\rho(\alpha); \sigma(\vec{\lambda})) = \frac{1}{d}$ written out as $\sum_i c_i(\alpha)\lambda_i = 0$, with

$$c_i(\alpha) = \frac{\text{tr}[(\rho(\alpha) \otimes P_i)(\mathbb{P}_{A_{0,1}} \otimes \mathbb{P}_{B_{0,1}} \otimes (d\mathbb{P} - \mathbf{1})_2)]}{\text{tr}[P_i]}$$

is linear in λ_i and thus defines a plane separating symmetric states activating $\rho(\alpha)$ from states apparently not activating it. The task is now to construct this separating plane depending on the parameter α.

As we have already used above, the points $\vec{\tau}^{(3)}, \vec{\tau}^{(4)}$ correspond to product states of the form $\sigma = \rho(\alpha)_1 \otimes \omega(\frac{1}{d})_2$ for which $f(\rho(\alpha); \sigma) = \frac{1}{d}$ obviously holds for any Werner state $\rho(\alpha)$. Thus $\vec{\tau}^{(3)}, \vec{\tau}^{(4)}$ are two fixed points of the separating plane with respect to a variation of α, and we need to know only one more point. For this purpose we consider the line $\vec{\lambda}(t) = t\vec{\tau}^{(5)} + (1-t)\vec{\tau}^{(1)}$. Solving the equation $f(\rho(\alpha); \sigma(\vec{\lambda}(t))) = \frac{1}{d}$ yields the required third point with

$$t = \frac{2+d}{2\alpha + d}. \tag{9}$$

This is obviously a strict monotone function in α, and it leads to the following properties of the separating plane.

(1) For $\alpha = \frac{d}{2}$ corresponding to the boundary Werner state which is not 1-distillable, Eq. (9) leads to $\vec{\lambda}(t) = \vec{\tau}^{(0)}$, showing that $\vec{\tau}^{(0)}$ indeed lies on the boundary of the set of separable states. That is, any PPT state in front of the plane $\{\vec{\tau}^{(0)}, \vec{\tau}^{(3)}, \vec{\tau}^{(4)}\}$ must be bound entangled since it activates at least $\rho(\alpha = \frac{d}{2})$.

(2) In the limit $\alpha \to 1$, i.e., $\rho(\alpha)$ becoming less and less entangled, $\vec{\lambda}(t)$ approaches $\vec{\tau}^{(5)}$. However, for any $\alpha = 1 + \varepsilon, \varepsilon > 0$ the polytope $\{\vec{\tau}^{(0)}, \vec{\tau}^{(3)}, \vec{\tau}^{(4)}, \vec{\lambda}(t)\}$ has a nonempty interior corresponding to PPT bound entangled states capable of activating any $\rho(\alpha)$ with $\alpha > 1 + \epsilon$.

(3) Except for the line $\{\vec{\tau}^{(3)}, \vec{\tau}^{(4)}\}$ all PPT states on the plane $\{\vec{\tau}^{(3)}, \vec{\tau}^{(4)}, \vec{\tau}^{(5)}\}$ lie on the activating side of the separating plane for any $\alpha > 1$. The corresponding symmetric states can thus be considered to be *universal activators* in the sense that they activate any entangled Werner state and therefore any NPPT state.

The set of bound entangled *universal activators* contains states arbitrarily close to the line $\{\vec{\tau}^{(3)}, \vec{\tau}^{(4)}\}$ which in turn corresponds to separable states. By the continuity properties of the entanglement measures *entanglement of formation* [17] and *relative entropy of entanglement* [18] this geometric vicinity, however, translates directly to the proposition that these states contain an arbitrary small amount of entanglement.

(iii) Bound entangled and not activating: In order to complete the classification of the symmetric states introduced in Eq. (3) we have still to determine the entanglement properties of the states corresponding to the tetrahedron $\{\vec{\tau}^{(0)}, \vec{\tau}^{(2)}, \vec{\tau}^{(4)}, \vec{\tau}^{(5)}\}$. The plane separating this set from the separable states derived above is characterized by a linear operator W via $\text{tr}[W\sigma(\vec{\lambda})] = 0$, where

$$W = (\mathbf{1} - \mathbb{F})_1 \otimes \left(\mathbf{1} - \frac{d}{2}\mathbb{P}\right)_2. \tag{10}$$

However, this operator is an *entanglement witness* (cf. [19]), meaning that $\text{tr}[W\rho] \geq 0$ holds for any separable state ρ. In order to see this property we have just to utilize the results from [4,6], where it was shown that $\mathbf{1} - \frac{d}{2}\mathbb{P}$ has a positive expectation value on any pure state of Schmidt rank two. Since the antisymmetric projector $\frac{1}{2}(\mathbf{1} - \mathbb{F})$ is a sum of Schmidt rank two states, $\langle\phi_A \otimes \psi_B|W|\phi_A \otimes \psi_B\rangle$ is a sum of such positive expectations for any pure product state $|\phi_A \otimes \psi_B\rangle$. Hence, $\text{tr}[W\rho] \geq 0$ is indeed fulfilled by any separable state implying that the tetrahedron under discussion corresponds to bound entangled symmetric states which are, however, not activating (with respect to the considered protocol).

Conclusion.—We investigated the entanglement properties of a new Abelian set of symmetric states with regard to the activation of NPPT distillation. The set contains PPT bound entangled states of full rank providing a universal resource for the activation of any NPPT state (in any finite dimension). Some of these *universal activators* lie arbitrarily close to separable states. Hence, the activation process turns out to require only an infinitesimal amount of entanglement. This indicates that the difference between 1-distillable and n-undistillable or even bound entangled NPPT states is very subtle. Moreover, the above results show that even weakly entangled bound entangled PPT states can be useful for some quantum information processing purposes [20].

As the problem discussed in this paper is primarily a feasibility problem, we have at this stage not asked for the obtained rates. In fact, one could, for instance, easily improve the probability of success for the used activation protocol by a factor of d^2 by measuring in a basis of maximally entangled states and retaining the state whenever the measurement outcomes coincide. An interesting question going one step further and requiring knowledge about rates is whether there is the possibility of *self-activation* after some initial activation with a limited resource took place. In other words, is it possible to yield asymptotically more entanglement from distillation than is needed for the preparation of the activator states?

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