Estimate of the Cosmological Bispectrum from the MAXIMA-1 Cosmic Microwave Background Map

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We use the measurement of the cosmic microwave background taken during the MAXIMA-1 flight to estimate the bispectrum of cosmological perturbations. We propose an estimator for the bispectrum that is appropriate in the flat sky approximation, apply it to the MAXIMA-1 data, and evaluate errors using bootstrap methods. We compare the estimated value with what would be expected if the sky signal were Gaussian and find that it is indeed consistent, with a χ^2 per degree of freedom of approximately unity. This measurement places constraints on models of inflation.

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Introduction.—All theories of structure formation in the universe predict the properties of the probability distribution function (PDF) of cosmological perturbations. In all cases of interest, the PDF can be completely described in terms of its spatial n-point correlation functions, which are the expectation values of all possible products of the random field with itself at different points in space. Under the assumption of statistical isotropy and homogeneity, it is normally more useful to characterize the PDF in terms of higher order moments of the Fourier transform of the field. Most readers are familiar with the two-point moment, the power spectrum of fluctuations (C_ℓ). Indeed current efforts in the analysis of cosmic microwave background (CMB) data have focused mainly on increasingly precise estimates of the angular power spectrum. The theoretical bias for this is clear: For inflation induced perturbations, which is the current favorite model of structure formation, the statistics are Gaussian and all nonzero moments of order $n > 2$ can be expressed in terms of the C_ℓ.

In this Letter, we present the first estimate of the bispectrum of the CMB on degree, and subdegree, angular scales. The bispectrum is the cubic moment of the Fourier transform of the temperature field, and it can be seen as a scale dependent decomposition of the skewness of the fluctuations (in much the same way as the C_ℓ is a scale dependent decomposition of the variance of fluctuations). The bispectrum can be used to look for the presence of a non-Gaussian signal in the CMB sky. We use the data collected with the MAXIMA-1 experiment [1] to quantify the bispectrum of the CMB. The Gaussianity of this data set has already been analyzed using complementary methods in [2], including the methods of moments, cumulants, the Kolmogorov test, the χ^2 test, and Minkowski functionals in eigen, real, Wiener-filtered, and signal-whitened spaces.

In the past few years, interest in the bispectrum has grown in the scientific community. Estimates of the bispectrum in the COBE data proved the statistic to be extremely sensitive to some non-Gaussian features in the data, be they cosmological or systematic [3]; the quality of galaxy surveys has made it possible to test for the hypothesis that the matter overdensity is a result of nonlinear gravitational collapse of Gaussian initial conditions [4]. On the other hand, a serious effort has been undertaken to calculate the expected bispectrum from various cosmological effects; secondary anisotropies (such as the Ostriker-Vishniac effect, lensing, Sunyaev-Zel'dovich effect) [5], as well as primordial sources (such as nonlinear corrections to inflationary perturbations or cosmic seeds) may lead to observable signatures in the bispectrum of the CMB [6–9].

Let us establish some notation. We shall be working in the small sky approximation where a map of the CMB can be considered approximately flat [10]. The anisotropy of the CMB, $\Delta T(\mathbf{x})$, can then be expanded in terms of two-dimensional Fourier modes as follows:

$$\Delta T(\mathbf{x}) = \int \frac{d^2k}{(2\pi)^2} a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{1}$$

As stated above, the complete statistical properties of ΔT can be encoded in the expectation values of products of the $a(\mathbf{k})$. The power spectrum is defined to be $\langle a(\mathbf{k}) a(\mathbf{k}')\rangle = (2\pi)^2 C(k)\delta^2(\mathbf{k} + \mathbf{k}')$. On small angular scales, the correspondence between the flat sky power spectrum and the full sky angular power spectum is straightforward: $C_\ell = C(k)|_{k=\ell}$. The bispectrum is defined to be

$$\langle a(\mathbf{k}_1)a(\mathbf{k}_2)a(\mathbf{k}_3)\rangle = (2\pi)^2 B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \times \delta^2(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3), \quad (2)$$

where the delta function constraint is a consequence of the assumption of statistical isotropy.

Method.— In this Letter, we take the approach adopted by Ferreira, Magueijo, and Górski in the analysis of the COBE four-year data [3]: We construct an estimator for the bispectrum, apply it to the MAXIMA-1 data, and quantify its variance using Monte Carlo methods. The MAXIMA-1 experiment and dataset is described in detail in Ref. [1]; as in [2] we use a map with square pixels of $8'$ each.

Given a map, we fast Fourier transform it and construct the following bispectrum estimator:

$$\hat{B}_{\ell_1\ell_2\ell_3} = \frac{1}{N_{\ell_1,\ell_2,\ell_3;\Delta_\ell}} \sum_{\mathbf{k}_i \in S_{(\ell_i,\Delta\ell)}} \mathrm{Re}[a(\mathbf{k}_1)a(\mathbf{k}_2)a(\mathbf{k}_3)], \quad (3)$$

with

$$\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0, \quad (4)$$

where $S_{(\ell_i,\Delta\ell)}$ is a ring in Fourier space centered at $\mathbf{k} = 0$ and with radial coordinates $k \in [\ell_i - \Delta_\ell/2, \ell_i + \Delta_\ell/2]$, $N_{\ell_1,\ell_2,\ell_3;\Delta_\ell}$ are the number of modes which satisfy this condition, and $\mathrm{Re}\{A\}$ is the real part of A. For a given choice of ℓ_i (with $i = 1, 2, 3$), we obtain an estimate of the bispectrum averaged in a bin of width Δ_ℓ. We correct for the finite resolution of the experiment and the pixelization of the map by replacing the quantity $a(\mathbf{k})$ (that is estimated directly from the map) by $a(\mathbf{k})/[B(\mathbf{k})W(\mathbf{k})]$, where $B(\mathbf{k})$ and $W(\mathbf{k})$ are the beam and pixel window functions, respectively (see [11] for a detailed Fourier space description of the beam).

There are a number of approximations in our analysis. We do not discuss any systematic effects that may have come into play when generating the map; a detailed description of these effects is presented in [12]. The flat sky approximation in the estimate of the power spectrum is valid to within 1% for the MAXIMA-1 100-square-degrees map. The fact that we are not considering a full sky map leads to two further complications [13]. First, there will be a finite correlation length in Fourier space between adjacent modes. In maximum-likelihood methods, this is automatically taken into account when constructing the correlation matrix, but in our case we must take care in assessing how our results depend on the width of the bins, Δ_ℓ, in which we estimate our bispectrum. Second, the map we are working with does not have periodic boundary conditions, an essential underlying assumption when performing a fast (or discrete) Fourier transform. We correct for this by multiplying the map by a Welch window function which suppresses the mismatch at the border of the map, thus reducing the leakage between neighboring scales in Fourier space. Naturally, we take this into account when estimating the bispectrum. Finally, the map to which we apply our estimator will contain anisotropic instrumental noise, and one may be concerned that this may bias the estimate. However, given that the signal and noise are uncorrelated, and the noise is Gaussian [12], it will not affect our estimate of the bispectrum, merely its variance.

Our goal in this Letter is twofold. First, to obtain an estimate of the bispectrum from the data without making any assumptions about the statistics of the signal, and, second, to assess how compatible our estimate of the bispectrum is with the assumption that the MAXIMA-1 data set is Gaussian. To obtain a model independent estimate of the bispectrum and its variance, we use bootstrap methods [14]. Bootstrap methods are widely used in situations where one wishes to extract the statistical properties of a given estimator without making any assumptions about the distribution from which a sample has been drawn.

One can redefine the estimator in Eq. (3) in the following way: Divide the ring in Fourier space into six equally sized angular segments of width $2\pi/6$; subdivide each of these segments into $M = 2\pi\ell/6\Delta_\ell$ angular slices. Within each of these slices apply the estimator in Eq. (3), replacing $S_{(\ell,\Delta\ell)}$ by the corresponding set of points within the slice. Note that this is applicable only to the diagonal components of the bispectrum $B_{\ell\ell\ell}$ (the inclusion of nondiagonal components will introduce correlations between samples which will bias bootstrap estimates). In this way, we find M approximately independent estimates of the $B_{\ell\ell\ell}$; note that $\Delta_\ell > 2\pi/(\text{field size})$ for this to be possible. If we find the average of these M estimates, we recover the value one obtains by applying (3). The fact that we have M (almost) independent estimates puts us in the condition where one can apply bootstrap methods to estimate the distribution and consequently the variance. We should note, however, that there are two limitations to this approximation. On the one hand, the sky signal is not uniformly distributed in Fourier space; i.e., there may be weak correlations between different Fourier modes. On the other hand, the noise is anisotropic and correlated, which means that the noise covariance matrix is not diagonal in Fourier space. Both of these effects may lead to correlations between the M approximately independent samples, but for large enough Δ_ℓ they should be negligible. Given that the bootstrap method is the only nonparametric (or model independent) method which one can apply in this situation, we choose to neglect these correlations [14].

Our approach to test for the Gaussianity is to generate 10^5 Monte Carlo realizations of the MAXIMA-1 data set, assuming a Gaussian signal with the power spectrum of the best fit model to the band powers estimated in [1]. Note that each of these mock data sets will have a realization of the noise which obeys the full anisotropic, nondiagonal correlation matrix; moreover, the effect of pixelization and finite beam are taken into account. We then compare our estimate of the real data with the Gaussian ensemble and quantify a goodness of fit.

Results.—We present the results we have obtained analyzing a square patch in the center of the MAXIMA-1 map, with 50^2 pixels. Given the dimensions of the map, we consider $\Delta_\ell = 75$; these correspond to the bin widths of the estimates of the C_ℓ in [1] and lead to correlations of the order of a few percent between adjacent bins. In Fig. 1 we present the diagonal elements ($\ell_1 = \ell_2 = \ell_3 = \ell$) of the estimate of the bispectrum (see also Table I). Note that all values of the bispectrum are of order $(0.001\text{–}0.01)C_\ell^{3/2}$, and the fact that $B_{\ell\ell\ell}|_{\ell=224}$ is so large is mostly due to the fact that this corresponds to the peak value of C_ℓ.

The boostrap errors are evaluated from resamplings with replacement of the approximately independent samples within each ring; the errors correspond to the 68% confidence regions with these simulated distributions. We find that the average bootstrap errors σ_{bs} over an ensemble of Gaussian maps to be between 4% and 8% lower than the true underlying variance. This bias is due to the correlations between adjacent samples within each ring. Moreover, the number of approximately independent samples ranges from $M = 2$ at $\ell = 148$ to $M = 10$ at $\ell = 748$, and one should therefore bear in mind that, for low ℓ, the variance in the estimate of the bootstrap errors is large.

We have performed a number of tests to evaluate how robust the result is on the parameters of our estimator. We have taken a larger patch of the MAXIMA-1 map and considered maps of 50^2 pixels with different locations within the MAXIMA-1 map. The estimated bispectra vary by a few percent. Alternatively, we have considered different bin widths (with $\Delta_\ell = 60$ and $\Delta_\ell = 90$) and found that estimates of the bispectrum vary smoothly and are consistent within different binnings. The use of the Welch window function turns out to be essential for small ℓ; this is to be expected as it should be the values of $B_{\ell\ell\ell}$ for low ℓ which are most affected by finite size effects. A different choice of window function (such as the Bartlett window function) changes the estimate of $B_{\ell\ell\ell}|_{\ell=148}$ and $B_{\ell\ell\ell}|_{\ell=224}$ by an order of 15% but leaves the remaining values of $B_{\ell\ell\ell}$ unaffected. One final test we have undertaken was to rotate the ring considered in Fourier space, this way displacing the M angular slices; we have found that the results vary by at most 10% in the lowest ℓ bin.

In Fig. 2, we plot the diagonal estimate of the MAXIMA-1 bispectrum compared to the 68% and 95% contour values if the sky was indeed Gaussian. We have checked that our statistic is unbiased even in the presence of anisotropic Gaussian noise and, as can be seen, the MAXIMA-1 B_ℓ seem to be consistent with the Gaussian assumption. The obvious way to quantify this is to use a goodness of fit. For the Monte Carlo realizations of the Gaussian sky signal, we find that most of the histograms of the $B_{\ell_1\ell_2\ell_3}$ are well approximated by Gaussians, and we therefore define the standard $\chi^2 = \sum_{\ell_1\ell_2\ell_3\ell_1'\ell_2'\ell_3'}(B^{\text{obs}}_{\ell_1\ell_2\ell_3} - B^{\text{th}}_{\ell_1\ell_2\ell_3})C^{-1}_{\ell_1\ell_2\ell_3\ell_1'\ell_2'\ell_3'}(B^{\text{obs}}_{\ell_1'\ell_2'\ell_3'} - B^{\text{th}}_{\ell_1'\ell_2'\ell_3'})$, where $B^{\text{th}}_{\ell_1\ell_2\ell_3} = 0$ and C is the covariance matrix of the estimators evaluated from the Monte Carlo realizations. In all, we have 115 values and we find $\chi^2 = 130$. From 10^4 realizations, we construct the expected distribution of this χ^2: We find that 70% of the distribution is contained to the left of the measured value. Even if we remove the outlier from the set of bins centered at $\ell = 224$ we still find that 52% of the distribution lies to the left of the measured χ^2.

Cosmological implications.—One can roughly divide the two possible sources of nonGaussianity in the CMB into primordial and late time. The latter have been extensively studied in [5] and typically give rise to nonzero bispectra on very small angular scales ($\ell > 1000$). We do not expect to find any evidence for such signatures in the MAXIMA-1 map. Moreover, the observed bispectrum limits the point source contribution to the MAXIMA power spectrum as it shows no significant rise at high ℓ. Primordial effects may give rise to non-Gaussianity on degree scales, and we shall focus on a few possibilities now. Inflation predicts almost Gaussian fluctuations to a very good approximation; there is, however, the possibility that second order corrections in the evolution of the inflaton field

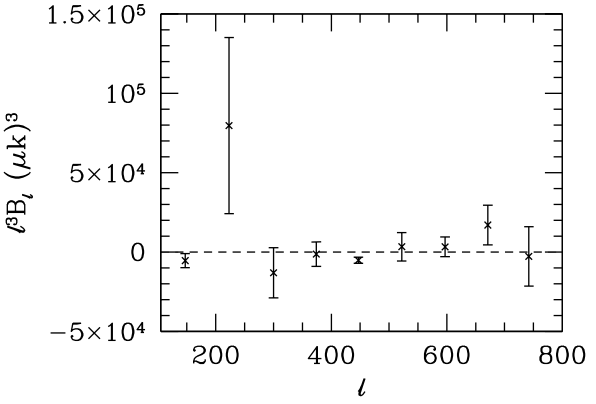

FIG. 1. Estimate of the bispectrum of the MAXIMA-1 CMB map. The error bars are evaluated using a Monte Carlo bootstrap method. Note that, given the small number of samples for the low ℓ components, there is a large uncertainty in the estimation of the error bars.

TABLE I. Measured bispectrum values and corresponding errors. The first column has the bandwidths, the second column the estimate of the bispectrum, the third column has an estimate of its variance using bootstrap methods, the fourth column has an estimate of its variance assuming the signal is Gaussian, and the fifth column has its variance just due to noise. Columns 2-5 are in units of $(\mu\text{K})^3$.

$[\ell_{\min}, \ell_{\max}]$	$\ell^3 B_\ell$	$\ell^3 \sigma_{bs}$	$\ell^3 \sigma_G$	$\ell^3 \sigma_N$
[111, 185]	−5455	4477	16 329	38
[186, 260]	79 622	55 440	41 363	145
[261, 335]	−13 167	15 798	17 590	183
[336, 410]	−1373	7687	8504	366
[411, 485]	−5208	1977	7593	1071
[486, 560]	3298	8939	8801	1815
[561, 635]	3199	6213	9387	2892
[636, 710]	16 952	12 518	13 997	5939
[711, 785]	−2802	18 725	26 058	14 197

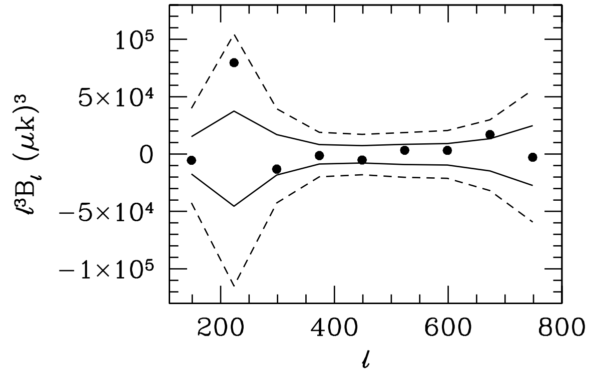

FIG. 2. Comparison with a Gaussian sky. The solid (dashed) lines delimit the 68% (95%) confidence region determined from a Monte Carlo simulation of a Gaussian sky; the MAXIMA-1 noise covariance matrix was used to simulate realistic, anisotropic noise, and the beam and pixel window functions were included.

may lead to mild non-Gaussianity. Komatsu and Spergel [7] have parametrized this nonlinearity in terms of a "nonlinear coupling constant," f_{NL}, which can be related to dynamical parameters in a variety of models of inflation. For example, $f_{NL} \simeq (3\epsilon - 2\eta)$, where ϵ and η are the slow roll parameters of single field inflation; one expects from slow roll models that at most $f_{NL} \simeq \mathcal{O}(1)$. An order of magnitude estimate gives

$$B_{\ell_1\ell_2\ell_3} \simeq b_{\ell_1\ell_2} + b_{\ell_2\ell_3} + b_{\ell_3\ell_1}$$
$$b_{\ell_i\ell_j} = -\frac{1.1\times 10^2}{T_{\rm CMB}} f_{NL}\left(\frac{\Delta T_{\ell_i}}{\ell_i}\right)^2\left(\frac{\Delta T_{\ell_j}}{\ell_j}\right)^2,$$

and $\Delta T_\ell^2 \equiv \ell(\ell+1)C_\ell/(2\pi)$. Using the Monte Carlo realizations described previously, it is possible to estimate the smallest amplitude, $|f_{NL}|$, distinguishable from the Gaussian hypothesis; we find that $|f_{NL}| < 944$ is indistinguishable from a Gaussian signal at the 95% confidence level. Note that the use of lower multipoles (as measured by COBE) should narrow this interval. A fit to the measured values using the Gaussian covariance matrix gives $|f_{NL}| \simeq 900$ ($\chi^2 = 122$).

More exotic possibilities can be considered, such as, for example, global topological defects. A semianalytic framework exists which allows one to calculate the statistical effects using the $O(N)$ nonlinear σ model. Different values of N will correspond to different types of localized objects, with, for example, $N = 2$ corresponding to global strings, $N = 3$ monopoles, and $N = 4$ corresponding to textures (taking N to infinity we recover Gaussianity). Verde *et al.* [8] (see also [6]) have estimated the bispectrum and found that

$$B_{\ell_1\ell_2\ell_3} \simeq \frac{2.0\times 10^5}{T_{\rm CMB}}\,\alpha\left(\frac{\Delta T_{\ell_1}}{\ell_1}\,\frac{\Delta T_{\ell_2}}{\ell_2}\,\frac{\Delta T_{\ell_3}}{\ell_3}\right)^{4/3},$$

where $\alpha = N^{-1/2}$ (we should point out that this expression was derived for large angles). In what follows, we shall extrapolate this expression to subdegree scales. The current sensitivity is such that models with $\alpha \simeq {<}2.4$ are indistinguishable from Gaussian theories; this range of α corresponds to any value of N. We find the best fit α to be $\alpha = 2.2$. Current estimates of the bispectrum do not therefore constrain global topological defects.

The bispectrum analysis of the MAXIMA data indicates that the data is consistent with Gaussianity. This reinforces the conclusions obtained in [2] and validates the assumptions that go into the data-analysis pipeline, namely, the assumption of Gaussianity of the sky signal which goes into both maximum-likelihood and Monte Carlo estimates of the power spectra.

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