Near-Horizon Conformal Symmetry and Black Hole Entropy

S. Carlip*

Department of Physics, University of California, Davis, California 95616

(Received 4 March 2002; published 31 May 2002)

Near an event horizon, the action of general relativity acquires a new asymptotic conformal symmetry. For two-dimensional dilaton gravity, this symmetry results in a chiral Virasoro algebra, and Cardy's formula for the density of states reproduces the Bekenstein-Hawking entropy. This lends support to the notion that black hole entropy is controlled universally by conformal symmetry near the horizon.

DOI: 10.1103/PhysRevLett.88.241301 PACS numbers: 04.70.Dy, 11.25.Hf

Introduction.—Since the seminal work of Bekenstein [1] and Hawking [2], we have understood that black holes are thermodynamic objects, with characteristic temperatures and entropies. The Bekenstein-Hawking entropy depends on both Planck's constant $\hbar$ and Newton's gravitational constant G, and offers one of the few known "windows" into quantum gravity. In particular, the microscopic statistical mechanics of black hole thermodynamics may tell us a good deal about the fundamental quantum degrees of freedom of general relativity.

Until quite recently, standard derivations of black hole entropy involved only macroscopic thermodynamics, and a statistical mechanical description was more a hope than a reality. Today, in contrast, we face the opposite problem: we have many candidate descriptions of black hole statistical mechanics, all of which yield the same entropy despite counting very different states. In particular, there are two string theoretical descriptions, one that counts D-brane states [3] and another involving a dual conformal field theory [4], an approach in loop quantum gravity that counts spin network states [5], and a slightly more obscure method [6] based on Sakharov's old idea of induced gravity [7]. The problem of "universality" is to explain why these approaches agree, and why they agree with the original semiclassical computations [2,8] that know nothing of the details of quantum gravity.

One possible answer is that black hole thermodynamics may be controlled by a symmetry inherited from the classical theory. This idea has its roots in an observation [9,10] that black hole entropy in three spacetime dimensions can be obtained from Cardy's formula [11,12] for the density of states of a two-dimensional conformal field theory at the "boundary" of spacetime. A number of authors have tried to extend such arguments to black holes in arbitrary dimensions [13–24], but while these calculations seem to have the right "flavor," none is yet fully satisfactory [25–29]. In particular, all proposals so far require awkward boundary conditions at the horizon, and most fail in two spacetime dimensions, where there does not seem to be enough room at the horizon for the required degrees of freedom.

In this paper, I point out three new ingredients that lead to an improved description of the near-horizon symmetries of a black hole, and show how they may overcome these difficulties.

1. Conformal symmetry: In the presence of a black hole with a momentarily stationary region near its horizon, the Einstein-Hilbert action of general relativity acquires a new conformal symmetry. Let Δ be a segment of such a horizon (see Fig. 1), with a "momentarily stationary" neighborhood $\mathcal{N}$ admitting a Killing vector χ^a for which Δ is a Killing horizon. If f is a smooth function that vanishes outside $\mathcal{N}$, then under the transformation

$$g_{ab} \rightarrow \nabla_c(f\chi^c)g_{ab} \tag{1}$$

the action in n dimensions transforms, up to possible boundary terms, as

$$\begin{aligned}\delta I &= \frac{1}{16\pi G}\int_N \nabla_c(f\chi^c)g^{ab}G_{ab}\epsilon \\ &= \frac{1}{16\pi G}\frac{n-2}{2}\int_N f\chi^c\nabla_c R\epsilon = 0,\end{aligned} \tag{2}$$

where ϵ is the volume form and the last equality follows from the fact that χ^a is a Killing vector.

For (1) to be a genuine symmetry, it must preserve the relevant space of fields; that is, the new metric must also admit a Killing vector in $\mathcal{N}$. This will be the case if

$$(\chi^a\nabla_a)^2 f = 0. \tag{3}$$

Below, we shall generalize this argument to the case of an asymptotic symmetry, for which $(\chi^a\nabla_a)^2 f$ goes to zero at Δ.

2. Horizon symplectic form: In the presence of a horizon, the canonical symplectic form of general relativity picks up a new contribution from the horizon. This is most easily seen in the covariant canonical formalism [30], in which the symplectic form Ω for a collection

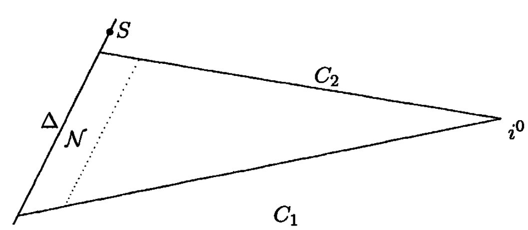

FIG. 1. Black hole spacetime: neighborhood $\mathcal{N}$ of horizon Δ, partial Cauchy surfaces C_1 and C_2, reference cross section S.

of fields ϕ is given by an integral $\Omega[\phi;\delta_1\phi,\delta_2\phi] = \int_C \omega[\phi;\delta_1\phi,\delta_2\phi]$ of a closed form ω over a (partial) Cauchy surface C. Consider the two surfaces C_1 and C_2 of Fig. 1. Since ω is closed,

$$\Omega_{C_1}[\phi;\delta_1\phi,\delta_2\phi] = \Omega_{C_2}[\phi;\delta_1\phi,\delta_2\phi] + \int_{\Delta\cap C_1}^{\Delta\cap C_2} \omega[\phi;\delta_1\phi,\delta_2\phi], \quad (4)$$

where the integral on the right-hand side is over the portion of the horizon joining C_1 and C_2. For the "isolated horizon" boundary conditions of Ref. [31], the restriction of ω to Δ is exact, and the horizon integral can be absorbed into Ω. In general, though, there is no reason to expect such simplicity. Instead, to define a symplectic structure that is independent of the Cauchy surface C, one must choose a "reference" cross section S of the horizon and define

$$\hat{\Omega}_C[\phi;\delta_1\phi,\delta_2\phi] = \int_C \omega[\phi;\delta_1\phi,\delta_2\phi] + \int_S^{\Delta\cap C} \omega[\phi;\delta_1\phi,\delta_2\phi], \quad (5)$$

where the second integral is over the portion of the horizon connecting S and C.

3. Asymptotic symmetry: The horizon of a generic black hole need not have a stationary neighborhood $\mathcal{N}$. The boundary conditions of Ref. [31], for example, require a Killing vector only on the horizon. What we need is a notion of an asymptotic symmetry, in which the spacetime is "almost" stationary as one approaches the horizon.

Traditionally, an "asymptotic symmetry" in general relativity has meant an exact symmetry that preserves some extra asymptotic structure. Here we have a different situation, a symmetry that may be exact only at the horizon, but that can be made arbitrarily good by shrinking the neighborhood $\mathcal{N}$. This is best viewed as a weakly broken symmetry. We can find an approximate Killing vector χ^a near the horizon (e.g., in the manner of [32]) and a metric $\bar{g}$ for which χ^a is an exact Killing vector, and write $g = \bar{g} + h$, where $h = 0$ at the horizon. The Lagrangian $\mathbf{L}[\bar{g} + h]$ is then invariant up to terms of order h, and the would-be Noether current for the transformation (1) is conserved up to terms of order h. While more work is needed to fully understand this sort of symmetry, it is evident that if h is smooth, an asymptotic symmetry near the horizon should become an exact symmetry for fields on the horizon itself.

The two-dimensional black hole.—We can now ask whether the new symmetry (1) places any restrictions on black hole thermodynamics. In general, one ought not to expect a symmetry to determine anything as "microscopic" as a density of states. There is one important exception, though: for a one- or two-dimensional conformal symmetry described by a Virasoro algebra with central charge c, the Cardy formula [11,12,33,34] tells us that the number of states having eigenvalue Δ of the "energy" L_0 goes asymptotically as

$$\rho(\Delta) \sim \exp\left\{2\pi\sqrt{\frac{c_{\text{eff}}\Delta}{6}}\right\}, \quad (6)$$

where $c_{\text{eff}} = c - 24\Delta_0$, with Δ_0 the lowest eigenvalue of L_0. This behavior is universal, holding independent of any details of what states are being counted.

The question is thus whether the symmetry (1) can be described by such an algebra. It is useful to focus on a particular example, two-dimensional dilaton gravity. This is not as restrictive as it may seem, since general relativity in any dimension can be dimensionally reduced via a Kaluza-Klein mechanism to two-dimensional gravity coupled to "matter" fields, and I shall argue below that the extra fields do not affect the conclusions. The action for dilaton gravity is [35]

$$I = \int \mathbf{L} = \frac{1}{2G}\int\left(\phi R + \frac{1}{L^2}V[\phi]\right)\epsilon, \quad (7)$$

where L is a coupling constant and V is an arbitrary function of the dilaton field ϕ. Strictly speaking, one cannot define the expansion of a null congruence in two dimensions, but the analog here is

$$\vartheta = (\ell^a\nabla_a\phi)/\phi, \quad (8)$$

where ℓ^a is the null normal. All known exact black hole solutions, including dimensionally reduced descriptions of higher-dimensional black holes, have null horizons with vanishing ϑ.

As in previous work [13,14], we will start with a "stretched horizon," here a null surface $\tilde{\Delta}$ with null normal ℓ^a for which ϑ is small but nonzero. Near a true horizon, we can take ϑ to be a measure of how far we have "stretched" away; in the end, we will take the limit $\vartheta \to 0$. The vector ℓ^a determines a unique "orthogonal" null vector n^a, such that $\ell^a n_a = -1$. We extend n^a from $\tilde{\Delta}$ by requiring that $n^a\nabla_a n_b = 0$, from which it follows that

$$\nabla_a\ell_b = -\kappa n_a\ell_b, \qquad \nabla_a n_b = \kappa n_a n_b, \quad (9)$$

where κ is the "surface gravity." Note, though, that unlike a timelike or spacelike unit vector, a null normal does not have a fixed normalization: by rescaling $\ell^a \to f\ell^a$, one can change κ almost arbitrarily on a fixed null surface $\tilde{\Delta}$ [31],

$$\kappa \to \ell^a\nabla_a f + \kappa f, \qquad n^a\nabla_a f = 0. \quad (10)$$

Observe from (9) that

$$\nabla_a\ell_b + \nabla_b\ell_a = \kappa g_{ab}, \quad (11)$$

so ℓ_a is a conformal Killing vector. We shall see later that the natural scaling of ℓ_a leads to a surface gravity κ proportional to ϑ, so ℓ_a is an approximate Killing vector near the horizon.

The application of the transformation (1) to two dimensions is a bit tricky, both because of the new field ϕ and because the field equations of dilaton gravity differ from those of ordinary general relativity. In general, we should expect ϕ as well as g_{ab} to transform, and it is easy to check that under a transformation

$$\begin{aligned}\delta g_{ab} &= \nabla_c(f\ell^c)g_{ab} = (\ell^c\nabla_c f + \kappa f)g_{ab}\,,\\ \delta\phi &= (\ell^c\nabla_c h + \kappa h)\,,\end{aligned} \tag{12}$$

the Lagrangian (7) satisfies $\delta\mathbf{L} \sim \vartheta$. We thus have an asymptotic symmetry in the sense described above. For now, the relationship of f and h will remain unspecified; we shall see later that the choice that makes the transformation (12) canonical implies that $\delta\mathbf{L} \sim \vartheta^2$.

Equation (12) is not enough to determine the separate variations of ℓ^a and n^a. This is to be expected, since the normalization of ℓ^a is not fixed; the only restriction, from (9), is that $n^a\nabla_a(n_b\delta\ell^b) = 0$. We are thus free to choose $\delta\ell^a = 0$, which then implies that

$$\begin{aligned}\delta\kappa &= \ell^b\nabla_b(\ell^c\nabla_c f + \kappa f)\,,\\ \delta s &= \ell^a\nabla_a(\ell^b\nabla_b h + \kappa h)\,,\end{aligned} \tag{13}$$

where $s = \ell^a\nabla_a\phi = \vartheta\phi$. It follows that

$$\begin{aligned}&[\delta_1,\delta_2]g_{ab} = (\ell^c\nabla_c\{f_1,f_2\} + \kappa\{f_1,f_2\})g_{ab}\\ &\text{with } \{f_1,f_2\} = (\ell^a\nabla_a f_1)f_2 - (\ell^a\nabla_a f_2)f_1\,,\end{aligned} \tag{14}$$

giving the standard conformal algebra.

To express the transformations (12) in Hamiltonian form, we need the symplectic form $\hat\Omega$ of (5). This can be computed by Wald's methods [30,35]. For variations that have their support only in a small neighborhood $\mathcal{N}$ of $\tilde\Delta$, the main contribution will come from the integral along $\tilde\Delta$. Restricting the symplectic form of Ref. [35] to $\tilde\Delta$, one finds that

$$\begin{aligned}\hat\Omega &= \frac{1}{2G}\int_{\tilde\Delta}[\ell^a\nabla_a(\delta_1\phi)\ell_b\delta_2 n^b - \ell^a\nabla_a(\delta_2\phi)\ell_b\delta_1 n^b]\hat\epsilon\\ &= -\frac{1}{2G}\int_{\tilde\Delta}(\delta_1\phi\,\delta_2\kappa - \delta_2\phi\,\delta_1\kappa)\hat\epsilon\,,\end{aligned} \tag{15}$$

where $\hat\epsilon = n$ is the induced volume element on $\tilde\Delta$ and the last equality follows from (13).

Hamiltonian and Virasoro algebra.—The next question is whether the transformation (12) is canonical, that is, whether it is generated by a "Hamiltonian" L. Such a Hamiltonian must satisfy [30]

$$\begin{aligned}\delta L[f,h] &= \hat\Omega[\delta,\delta_{f,h}]\\ &= -\frac{1}{2G}\int_{\tilde\Delta}[\delta\phi\,\ell^b\nabla_b(\ell^c\nabla_c f + \kappa f)\\ &\qquad - \delta\kappa(\ell^c\nabla_c h + \kappa h)]\hat\epsilon\,,\end{aligned} \tag{16}$$

where again $s = \ell^a\nabla_a\phi$. The variation δ is an exterior derivative on the space of fields, and the integrability condition for (16) is that $\delta^2 L[f,h] = 0$. If we assume that the parameters f and h are field independent, this condition requires that δs be proportional to $\delta\kappa$. In particular, this proportionality must hold for variations of the form (12), and this, together with the requirement that $\delta f = \delta h = 0$, implies that

$$\frac{\kappa}{s} = \text{const on } \tilde\Delta, \qquad sf = \kappa h\,. \tag{17}$$

Despite appearances, (17) is not a real restriction on the geometry, since κ/s can always be rescaled to a constant on $\tilde\Delta$ using (10). As noted earlier, this relation makes the transformation (12) an even better approximate symmetry. Indeed, the variation of the "kinetic term" ϕR in the action now goes as ϑ^2 near Δ, and the variation of the "potential term" can also be arranged to be of this order by a suitable choice of κ/s on $\tilde\Delta$.

With the relation (17) between h and f, (16) can easily be integrated, yielding

$$\begin{aligned}L[f] &= \frac{1}{2G}\int_{\tilde\Delta}s(2\ell^a\nabla_a f + \kappa f)\hat\epsilon\\ &= -\frac{1}{2G}\int_{\tilde\Delta}(2\ell^a\nabla_a s - \kappa s)f\hat\epsilon\,.\end{aligned} \tag{18}$$

We must next choose a basis for the functions f on $\tilde\Delta$. Since the normalization of ℓ^a is not fixed, the corresponding light cone coordinate has no intrinsic physical meaning. There is, however, a natural coordinate on $\tilde\Delta$, the dilaton ϕ itself, which by the two-dimensional version of the Raychaudhuri equation should be monotonic on $\tilde\Delta$. Let

$$z = e^{2\pi i\phi/\phi_+}\,, \tag{19}$$

where ϕ_+ is the value of ϕ on the horizon, so $z \to 1$ at Δ [36]. We can then choose a basis of functions to be proportional to z^n, with the proportionality constants determined by (14):

$$f_n = \frac{\phi_+}{2\pi s}z^n, \qquad \{f_m,f_n\} = i(m-n)f_{m+n}\,. \tag{20}$$

Note that the consistency condition (3) is satisfied asymptotically: $(\ell^a\nabla_a)^2 f_n \sim \vartheta$ near Δ.

In terms of these modes, the Hamiltonian (18) becomes, on shell,

$$L[f_n] = -\frac{1}{2G}\frac{\kappa}{s}\frac{{\phi_+}^2}{2\pi}\delta_{n0}\,. \tag{21}$$

The Poisson brackets $\{L[f_m],L[f_n]\}$ can be computed directly from Eq. (16):

$$\{L[f_m],L[f_n]\} = \delta_{f_m}L[f_n] = -\frac{2\pi i}{G}\frac{s}{\kappa}n^3\delta_{m+n,0}\,, \tag{22}$$

which may be recognized as the expression for a central term in the Virasoro algebra with central charge

$$c = -\frac{24\pi}{G}\frac{s}{\kappa}\,. \tag{23}$$

We can now insert (21) and (23) into the Cardy formula (6). With the assumption that $\Delta_0 \approx 0$—i.e., from (21), that quantum black hole states extend all the way down to

$\phi_+ \approx 0$—we have $c_{\rm eff} \approx c$, yielding a density of states

$$\log\rho(L_0) = \frac{2\pi\phi_+}{G}. \tag{24}$$

This is precisely the Bekenstein-Hawking entropy for the two-dimensional dilaton black hole [35]. The central charge c used here is the "classical" central charge, but by the correspondence principle, it should give the leading $(O(1/\hbar))$ contribution to any quantum theory that has the correct classical limit. Additional quantum contributions may occur, but they will give corrections of order 1, unimportant for large black holes. It is worth emphasizing once again that the Cardy formula tells us nothing about *what* states are being counted. The relevant states need not even be "gravitational": the gravitational action determines the conformal symmetry, but the mere existence of this symmetry is enough to fix the asymptotic density of states.

In contrast to previous work on Virasoro algebras at the horizon, this derivation has the nice feature that the central charge (23) does not depend on the particular black hole being considered. The algebra may therefore be viewed as a universal one, with different black holes represented by different values of L_0. An extension of this analysis to higher dimensions would clearly be of interest. As noted above, though, higher-dimensional general relativity may be dimensionally reduced to two-dimensional dilaton gravity coupled to extra matter fields (see, for example, [37]). It is fairly easy to see that the added terms cannot contribute to the classical central charge (23), though they might give quantum corrections.

We should probably also worry further about making the notion of "asymptotic symmetry" used here more rigorous. It may be useful to exploit a generalization of the symmetry (1) that exists in the presence of a *conformal* Killing vector η^a, $\nabla_a\eta_b + \nabla_b\eta_a = \kappa g_{ab}$. It is not hard to check that the transformation

$$g_{ab} \rightarrow \left(\eta^c\nabla_c f + \frac{n-2}{2}\kappa f\right)g_{ab} \tag{25}$$

leaves the Einstein-Hilbert action invariant provided f is chosen to satisfy $\int g^{ab}\nabla_a\kappa\nabla_b f\epsilon = 0$. If the original metric g_{ab} admits a conformal Killing vector, the transformed metric does as well. Maintaining the condition on f is trickier, but at least one solution exists: if both κ and f are functions of a single null coordinate v, this restriction holds automatically. Work on understanding this extended symmetry is in progress.

This work was supported in part by Department of Energy Grant No. DE-FG03-91ER40674.

*Email address: carlip@dirac.ucdavis.edu

[1] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973).
[2] S. W. Hawking, Nature (London) **248**, 30 (1974).
[3] A. Strominger and C. Vafa, Phys. Lett. B **379**, 99 (1996).
[4] O. Aharony *et al.,* Phys. Rep. **323**, 183 (2000).
[5] A. Ashtekar *et al.,* Phys. Rev. Lett. **80**, 904 (1998).
[6] V. P. Frolov and D. V. Fursaev, Phys. Rev. D **56**, 2212 (1997).
[7] A. D. Sakharov, Sov. Phys. Dokl. **12**, 1040 (1968).
[8] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2752 (1977).
[9] A. Strominger, J. High Energy Phys. **9802**, 009 (1998).
[10] D. Birmingham *et al.,* Phys. Lett. B **424**, 275 (1998).
[11] J. L. Cardy, Nucl. Phys. **B270**, 186 (1986).
[12] H. W. J. Blöte, J. L. Cardy, and M. Nightingale, Phys. Rev. Lett. **56**, 742 (1986).
[13] S. Carlip, Phys. Rev. Lett. **82**, 2828 (1999).
[14] S. Carlip, Classical Quantum Gravity **16**, 3327 (1999).
[15] S. Carlip, Phys. Lett. B **508**, 168 (2001).
[16] S. N. Solodukhin, Phys. Lett. B **454**, 213 (1999).
[17] R. Brustein, Phys. Rev. Lett. **86**, 576 (2001).
[18] S. Das *et al.,* Phys. Rev. D **63**, 024023 (2001).
[19] J. Jing and M.-L. Yan, Phys. Rev. D **63**, 024003 (2001).
[20] M. Cadoni and S. Mignemi, Phys. Rev. D **59**, 081501 (1999).
[21] M. Cadoni *et al.,* Phys. Rev. D **63**, 125021 (2001).
[22] D. J. Navarro *et al.,* Nucl. Phys. **B580**, 311 (2000).
[23] F. L. Lin and Y.-S. Wu, Phys. Lett. B **453**, 222 (1999).
[24] M. I. Park, hep-th/0111224.
[25] M. I. Park and J. Ho, Phys. Rev. Lett. **83**, 5595 (1999).
[26] M. I. Park and J. H. Yee, Phys. Rev. D **61**, 088501 (2000).
[27] S. Carlip, Phys. Rev. Lett. **83**, 5596 (1999).
[28] O. Dreyer *et al.,* Classical Quantum Gravity **18**, 1929 (2001).
[29] J. I. Koga, Phys. Rev. D **64**, 124012 (2001).
[30] R. M. Wald, Phys. Rev. D **48**, R3427 (1993).
[31] A. Ashtekar *et al.,* Phys. Rev. D **62**, 104025 (2000).
[32] R. A. Matzner, J. Math. Phys. (N.Y.) **9**, 1657 (1968).
[33] S. Carlip, Classical Quantum Gravity **15**, 3609 (1998).
[34] R. Dijkgraaf *et al.,* hep-th/0005003.
[35] J. Gegenberg *et al.,* Phys. Rev. D **51**, 1781 (1995).
[36] This choice is almost unique, in that ϕ_+ is the only natural quantity having the right dimension. In principle, though, we could choose $w = z^\alpha$; this would leave the central charge unchanged, but would shift L_0.
[37] J. H. Yoon, Phys. Lett. B **451**, 296 (1999).