Stability and Bifurcations of the Figure-8 Solution of the Three-Body Problem

J. Galán,* F. J. Muñoz-Almaraz, and E. Freire
Departamento de Matemática Aplicada II, Escuela Superior de Ingenieros, University of Seville, Camino de los Descubrimientos s/n, 41092 Sevilla, Spain

E. Doedel
Department of Computer Science, Concordia University, 1455 Boulevard de Maisonneuve O, Montreal, Quebec, Canada, H3G 1M8

A. Vanderbauwhede
Department of Pure Mathematics and Computer Algebra, University of Gent, Krijgslaan 281 B-9000 Gent, Belgium
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The stability properties of a recently discovered solution of the general three-body problem with equal masses and the shape of a figure 8 are analyzed as the masses are varied. It is shown by numerical continuation and the evaluation of the characteristic multipliers that the solution is stable only in a narrow mass interval. Other less symmetrical and unstable solutions with equal masses in the same homotopy class as the figure-8 orbit have been found. The branching behavior is also analyzed.

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The three-body problem in celestial mechanics is very easy to define but impossible to solve. Since the formulation of the law of gravitation by Newton, it has inspired the progress of several branches of mathematics, and its importance is now perceived as much in the mathematical advances generated by the attempts at its solution as in the actual problem itself.

However, despite its deceiving simplicity, it is a complicated nonlinear problem and only very few proofs of existence of explicit periodic solutions are known [1–3]. Two of them have been named after some of the most prominent mathematicians, namely the equilateral solution of Lagrange in which the three bodies are located at the vertices of an equilateral triangle that rotates with constant angular velocity, and the collinear solution of Euler where two of the bodies rotate at constant angular velocity in a circle while the third one is located at rest exactly in the center of the circle. There are also elliptic shaped versions of these solutions with varying angular velocity. On the other hand, from the numerical point of view, a lot of work has been done and an overwhelming amount of knowledge has been accumulated over the years [4,5].

Since the work of Poincaré [6] it is known that the dynamical behavior of the three bodies can be very complicated (unpredictable) and that the only hope to gain some understanding of the backbone of the dynamical system is to study the periodic solutions of the problem. A classical (and unproven) conjecture affirms that any bounded solution of a Hamiltonian system can be approximated by a periodic orbit.

The spectacular discovery of Chenciner and Montgomery [7] of the existence of a new solution of the three-body problem with equal masses in which all the bodies follow the same eight-shaped curve has brought great excitement to the dynamical system community. This solution was first predicted by Moore [8] in the context of braids in classical dynamics. The method of proof is based on variational arguments; after some reductions the action integral is minimized in a restricted set of symmetric arcs to prove the existence of a solution in which the three bodies of equal masses chase each other following a closed trajectory. However, the variational proof is unable to decide about the stability of the solution.

Simó [9] computed this remarkable solution numerically with great accuracy and announced *elliptic stability*; i.e., the nontrivial characteristic multipliers of the periodic orbit are on the unit circle. The precise values of the nontrivial characteristic multipliers (those which are not always equal to one) are given by $\mu_j = \exp(2\pi i \nu_j)$, with $\nu_1 = 0.008\,422\,72$, $\nu_2 = 0.298\,092\,53$. Note that the smallness of ν_1 indicates that the figure-8 solution is close to a bifurcation.

Simó also discovered similar solutions for the case of three bodies in the planar case [10] (up to 345 in number) and for the general N-body problem $(3 < N < 799)$ [9], and gave them the name of "choreographies," the defining property being that all bodies follow a single closed curve in phase space with a fixed delay. From the historical point of view the solution of Lagrange in 1772 can be considered as the first "choreography." It has taken more that 200 years to find the second one.

There has been some controversy about the stability properties of the real minimizer of the action. It is not clear whether the minimizer has to be elliptic or parabolic/hyperbolic. In Hamiltonian systems with 2 degrees of freedom, minimizing orbits are always unstable [11]; however, for higher dimensional systems there are counterexamples to this statement. It has been suggested [12] that there must exist a less symmetric and unstable solution that is the actual minimizer of the action for the three-body problem with equal masses. From the variational point of view, if the symmetry restriction is relaxed and one enlarges the space of arcs over which the action is minimized, then obviously the action will not increase. The actual minimizing orbit would be in the same homotopy class as the figure-8 orbit [12], but no longer a choreography.

The purpose of this Letter is to try to clarify the relation between the above-mentioned elliptic and hyperbolic solutions, by studying the bifurcation behavior of the figure-8 solution as the masses are varied.

We have applied a continuation scheme [13] taking the numerically computed figure-8 orbit as the starting solution. We follow a one-parameter family of solutions and monitor the stability and the appearance of new branches at the bifurcation points. This combination of local stability analysis and global path following is a valuable and complementary approach to numerical simulations.

The equations of motion of the three bodies under their mutual gravitational attraction are very easy to state:

$$\begin{aligned}
\ddot{\mathbf{x}}_1 &= -m_2 \frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3} - m_3 \frac{\mathbf{x}_1 - \mathbf{x}_3}{|\mathbf{x}_1 - \mathbf{x}_3|^3}, \\
\ddot{\mathbf{x}}_2 &= -m_1 \frac{\mathbf{x}_2 - \mathbf{x}_1}{|\mathbf{x}_1 - \mathbf{x}_2|^3} - m_3 \frac{\mathbf{x}_2 - \mathbf{x}_3}{|\mathbf{x}_2 - \mathbf{x}_3|^3}, \\
\ddot{\mathbf{x}}_3 &= -m_2 \frac{\mathbf{x}_3 - \mathbf{x}_2}{|\mathbf{x}_3 - \mathbf{x}_2|^3} - m_1 \frac{\mathbf{x}_3 - \mathbf{x}_1}{|\mathbf{x}_1 - \mathbf{x}_3|^3},
\end{aligned} \tag{1}$$

where $\mathbf{x}_i \equiv (x_i, y_i, z_i)$ are the coordinates of the ith body in $\mathbb{R}^3$, the dot means a time derivative, m_i is the mass of the ith body, and the universal constant of gravitation is taken to be unity.

These equations can be written as a system of 18 first order differential equations and have seven time independent conserved quantities, namely the Hamiltonian, the three components of the linear momentum $\mathbf{P} = \sum_{i=1}^{3} m_i \dot{\mathbf{x}}_i$, and the three components of the angular momentum $\mathbf{L} = \sum_{i=1}^{3} m_i \mathbf{x}_i \wedge \dot{\mathbf{x}}_i$. These constants of motion are a direct consequence of the autonomous character of the equations and their invariance under translations and rotations.

Additionally, the equations are invariant under the transformation $\mathbf{x} \rightarrow c\mathbf{x}$ and $t \rightarrow c^{3/2} t$ [14,15]. Because of this scaling property of the equations, there is a trivial continuation on the period; arbitrarily close to any solution there exists another solution which is just a scaled version of the previous one and with exactly the same stability. To remove this "trivial" continuation family, the period has been fixed along the continuation branch and has been taken equal to 2π in this work; the continuation parameter which is allowed to vary will be one of the masses of the bodies (m_1).

In a symmetrical dynamical system the result of the application of a continuous symmetry on a generic periodic solution is another periodic solution with identical stability properties; that means that the orbits are foliating manifolds whose dimensions are equal to the number of conserved quantities plus one. For instance, in the case of a Hamiltonian system with no other conserved quantity this is the well known "cylinder theorem" [1]. This equivalence relation between solutions is the key to the symmetry reduction of the problem. The dimension of the problem can be reduced by eliminating some variables with the help of the integrals of motion [15]. However, in this work we have followed an alternative approach; we will maintain the dimensionality of the problem and take into account the symmetries in a different way.

The idea is to modify the systems in an appropriate way and, at the same time, impose additional boundary conditions to "freeze" the effect of the symmetry. The output is a one parameter family of periodic solutions along the manifold. The additional terms have to be chosen such that the periodic solutions of the modified system are, in fact, solutions of the original one. This same idea is present in the classical proof of the Lyapunov center theorem. The theoretical details of this continuation scheme as well as its application to simple models can be found in Ref. [13], and can be summarized as follows.

The equations of motion of the three bodies in a Hamiltonian formalism can be written as

$$\dot{u}(t) = J\nabla H(u(t), \lambda), \tag{2}$$

where $u(t)$ is a state vector in $\mathbb{R}^{18}$ with the nine components of the positions (generalized coordinates) and the nine components of the velocities (generalized moments) of the three bodies, $H \in C^\infty(U)$ is the Hamiltonian of the three interacting bodies, $J = \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ -\mathbb{I} & \mathbb{O} \end{pmatrix}$ is the simplectic matrix, $\mathbb{I}$ and $\mathbb{O}$ are the nine-dimensional unity and zero matrices, U is an open set in $\mathbb{R}^{18}$, and λ is the parameter which is allowed to vary during the continuation process. Let us denote the six additional conserved quantities as $F_i \in C^1(U)$ $(i = 1, \ldots, 6)$.

For a *generic* orbit of the dynamical system, the vectors $\nabla H(u, \lambda)$ and $\nabla F_i(u, \lambda)$ $(i = 1, \ldots, 6)$ will be *linearly independent*. Let u_0 be a periodic orbit of Eq. (2) with period T_0 and $\lambda = \lambda_0$. Finding a periodic solution with the same period T_0 and a different value of λ is equivalent to solving the following boundary value problem:

$$\begin{aligned}
u'(t) &= T_0 \Bigg[J\nabla H(u(t), \lambda) + \alpha \nabla H(u(t), \lambda) \\
&\qquad + \sum_{i=1}^{6} \beta_i \nabla F_i(u(t), \lambda) \Bigg], \\
u(1) &= u(0),
\end{aligned} \tag{3}$$

provided that α and β_i $(i = 1, \ldots, 6)$ vanish. For convenience the time has been rescaled using T_0 as a scaling factor, such that the period equals one; also, we define $\beta = (\beta_1, \beta_2, \ldots, \beta_6)$.

The continuation problem can be stated as

$$p = u(1; p, \lambda, \alpha, \beta), \tag{4}$$

where $u(t; p, \lambda, \alpha, \beta)$ is the solution of Eq. (3) with $u(0) = p$ as the initial condition. Consider a solution

$u_0(t)$ such that $u_0(1) = u_0(0)$ and such that $\lambda = \lambda_0$, $\alpha = 0$, and $\beta = 0$.

Construct the function $G : \mathbb{R}^{18+1+7} \mapsto \mathbb{R}^{18+7}$ by adding seven additional boundary conditions.

$$G(p, \lambda, \alpha, \beta) = \begin{bmatrix} u(1; p, \lambda, \alpha, \beta) - p \\ [p - u_0(0)]^* J \nabla H(u_0(0), \lambda) \\ [p - u_0(0)]^* J \nabla F_1(u_0(0), \lambda) \\ \cdots \\ [p - u_0(0)]^* J \nabla F_6(u_0(0), \lambda) \end{bmatrix}. \quad (5)$$

The zeros of this function correspond to periodic solutions of the problem. The theoretical result that allows the continuation of a one-parameter family of periodic solutions is the following.

Theorem 1: *Let u_0 be a solution of (2) with $T = T_0$, $\lambda = \lambda_0$ for which the gradients of the seven time independent first integrals are linearly independent and whose monodromy matrix has* 1 *as an eigenvalue with geometric multiplicity equal to* 7. *Then, there exists a unique branch of solutions of the equation $G(p, \lambda, \alpha, \beta) = 0$ close to $(u_0(0), \lambda_0, 0, 0)$. Moreover, along this branch α and β vanish.*

The proof of this result is a direct application of the implicit function theorem and can be found in [13].

We are now ready to analyze the stability and bifurcation behavior of the figure-8 orbit. It turns out that this solution is *generic* and fulfills the hypothesis of the theorem; it is a periodic solution of Eqs. (1) with period that can be taken equal to 2π and $\lambda = m_1 = 1$, the gradients of the seven conserved quantities are linearly independent, and the eigenvalue one of the monodromy matrix has geometric multiplicity equal to seven. The first output of our continuation algorithm [16] are the nontrivial characteristic multipliers for $m_1 = 1$; we find $\mu_j = \exp(2\pi i \nu_j)$ with $\nu_1 = 0.008\,422\,7$ and $\nu_2 = 0.298\,092\,5$. The agreement with the results of Simó [9] is a good test of our method.

The results of the continuation of the figure-8 as one of the masses is varied in a very small scale is shown in Fig. 1. The L_2 norm of the solution is plotted as a function of the mass of one of the bodies. The solution labeled by A is the starting point of our calculation (the Moore-Chenciner-Montgomery solution). This planar orbit is plotted in real space in the upper panel of Fig. 2.

The solid curve in Fig. 1 corresponds to the stable region of the family of solutions that emanates from the figure-8; in that narrow mass interval ($\sim 10^{-5}$) the multipliers are elliptic. As the mass m_1 is *increased,* the family reaches a limit point (LP) and the branches return to lower values of m_1. Exactly at the turning point the solution loses stability and becomes hyperbolic. If we continue along this branch we come back to the case of all masses equal to one. This solution, labeled B in Fig. 1, which by construction is in the same homotopy class as the figure-8 solution, is hyperbolic, has less symmetry in the sense that it is no longer a choreography, and was also computed numerically by Simó [10]. The three bodies follow three slightly different figure-8 paths which are plotted in the lower panel of Fig. 2.

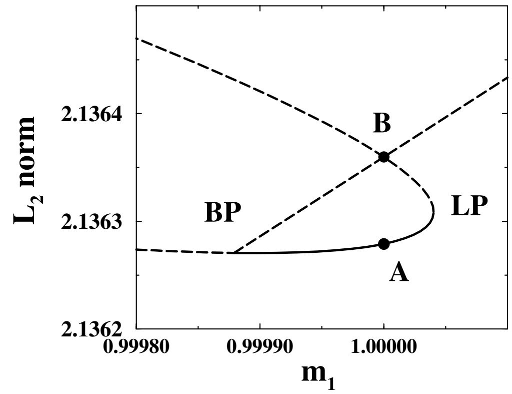

FIG. 1. Bifurcation diagram of the figure-8 solution. The stable region is a narrow window between a limit point (LP) and a branching point (BP) bifurcation.

However, if starting from solution A the mass m_1 is *decreased,* we reach a bifurcation point (BP) which corresponds to a symmetry breaking bifurcation in which two symmetry related branches are born (pitchfork bifurcation). When these solutions are continued until the value of $m_1 = 1$ is reached, the orbits are exactly the same as solution B but with an interchange of the role played by the bodies. The shape of the orbit is exactly the same as the one plotted in the lower panel of Fig. 2 but with a permutation of the labels of the bodies. The intersection of the three branches at B is just apparent; it is an artifact of the representation that we have chosen.

In Fig. 3 we shade the stability region in the m_1-m_2 plane of the figure-8 solution. It is straightforward to see that there are just two dimensionless parameters in Eq. (1) and, therefore, the third mass m_3 is kept fixed and equal to 1. The solid lines are the positions of the limit points. The branch points are labeled by BP and the position of the

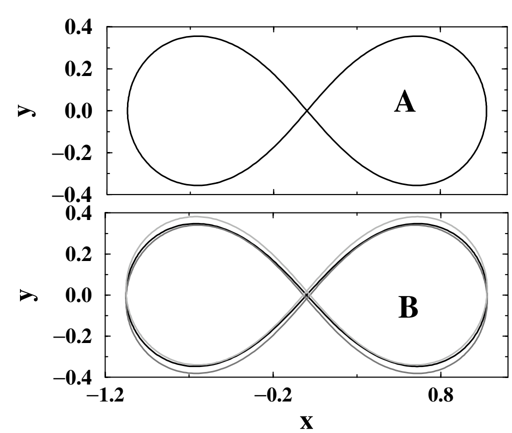

FIG. 2. Real space representation of the figure-8 solution (A) and the other unstable solution (B).

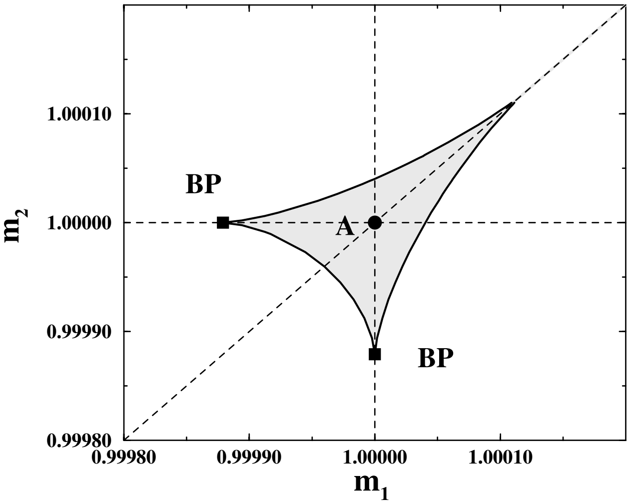

FIG. 3. Stability region in the m_1-m_2 plane of the figure-8 solution.

figure-8 solution is marked with an A. The dashed lines are just intended to visually enhance the symmetry along the diagonal.

In principle, the figure-8 solution could be continued into the restricted three-body problem along this diagonal by increasing the values of two of the masses, but in practice, we find that as the two masses are taken to be equal and large, then the bodies collide.

From the variational point of view, solution B is a minimizer of the action in a much larger space of functions than solution A. However, when the action is computed, we obtain the unexpected result that, within the precision of our calculations, *it has the same action as the figure-8 solution*. Using its standard definition, the value of the action integral is found to be $S = 24.371\,97$. This is not in contradiction with the variational principle but reveals a degeneracy that deserves further analysis.

There are obvious starting points to extend the continuation of the family of solutions far from the neighborhood of $m_1 = 1$, namely, the new branch that is born at the branching point and the unstable branch that persists after the B solution has been reached. These paths have been followed, and by continuing the subsequent branching of the solutions, it has been possible to connect the figure-8 solutions where all the masses are equal with the restricted three-body problem, where one of the masses vanishes. In between, we have found, as expected, a very rich bifurcation behavior, including collisions; these results will be presented elsewhere.

It is worth mentioning that all the solutions presented in this paper are *planar* but this is an outcome of the calculations. During the continuation process the orbits could develop spatial (three-dimensional) features but in the regions of this work they remain two-dimensional. The figure-8 solution is elliptically stable, but most of the other solutions found in the continuation process are unstable (hyperbolic). The continuation scheme has been designed to treat both solutions equally, but in real astronomical observations, only the stable periodic orbits are visible. By means of a careful analysis of the multipliers we have been able to detect other branches where the multipliers are elliptic. It would be interesting to apply our numerical scheme to the new recently discovered planetary systems HD168443 or Gliese 876 [17] to establish the mass interval in which stability can be preserved.

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*Electronic address: jgv@ma2.us.es

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