Interaction Cost of Nonlocal Gates

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We introduce the interaction cost of a nonlocal gate as the minimal time of interaction required to perform the gate when assisting the process with fast local unitaries. This cost, of interest both in the areas of quantum control and quantum information, depends on the specific interaction, and allows one to compare in an operationally meaningful manner any two nonlocal gates. In the case of a two-qubit system, an analytical expression for the interaction cost of any unitary operation given any coupling Hamiltonian is obtained. One gate may be more time consuming than another for any possible interaction. This defines a partial order structure in the set of nonlocal gates, that compares their degree of nonlocality. We analytically characterize this partial order in a region of the set of two-qubit gates.

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An elementary concern in quantum information theory is to establish the trade-off between different physical resources that are relevant for information processing. A controlled Hamiltonian interaction between quantum systems is one instance of useful resource. It can be employed, for example, to simulate the dynamics of another multipartite quantum system. On the other hand, multiparticle unitary gates are a requirement for universal quantum computation. In particular, two-qubit gates — together with one-qubit gates — can be taken as the building blocks of quantum computers.

A detailed study of the connections existing between nonlocal Hamiltonians and nonlocal gates is thus of interest from a quantum information perspective, but this issue is also relevant in other areas. For instance, the synthesis of multipartite gates from Hamiltonian interactions — and, in particular, time-minimizing schemes — has been recently analyzed in the context of quantum control theory [1]. Whereas the requirements for arbitrary manipulation of single qubits are presently met in a number of experimental schemes, the engineering of two-qubit gates can be only (partially) achieved with very few systems [2]. In real experiments not only an interaction Hamiltonian between the qubits but also considerable command of them in order to process the interaction are required. For instance, mechanisms to switch on and off the interaction, as well as to accurately drive the systems towards the desired joint evolution, are needed. But even from a theoretical perspective, a description of two-qubit gates in terms of interactions able to prescribe optimal protocols for gate synthesis was so far missing. In this Letter we shall provide such a description.

More generally, we consider a set of subsystems with a given Hamiltonian H, and assume that arbitrarily fast local unitaries (LU) can be performed to properly tailor the evolution that H induces. The aim is to perform some joint unitary transformation U on the systems. This is the setting considered in [1] and corresponds to the so-called *gate simulation under LU* of [3]. Two definitions are needed to specify the problems that we shall address.

Definition 1: The interaction cost $C_H(U)$ of a nonlocal gate U given a Hamiltonian H denotes the minimal time needed in order to perform U using the interaction H and fast LU.

Definition 2: We say gate U is more nonlocal than gate V, and write $V \leq U$, when for all interactions H the interaction cost of U is never smaller than that of V,

$$V \leq U \equiv C_H(V) \leq C_H(U) \quad \forall\, H\,. \tag{1}$$

First we shall show how the interaction cost C_H can be explicitly computed for any gate and any interaction of a two-qubit system. This is possible by considering the results recently developed in the areas of quantum control [1] and quantum information [3–6]. In [1] considerable progress towards the solution was made, and only a final optimization was left unsolved. The results of [3–6] provide the tools needed to perform such an optimization and thereby complete the results of [1].

Definition 2 introduces a partial order structure in the set of nonlocal gates. This structure captures the intuition, in terms of the resources needed to perform a gate, that one gate may be "more nonlocal" than another [7]. Our second result is an analytical characterization of this structure in a region of the set of two-qubit gates.

We start by describing known facts concerning the simulation of nonlocal Hamiltonians and the synthesis of nonlocal gates.

(*i*) *Optimal simulation of two-qubit Hamiltonians under LU.*— Any Hamiltonian acting on two qubits is uniquely represented, for the purposes of simulation under LU, by its canonical form [3,4]

$$H = \sum_i h_i \sigma_i \otimes \sigma_i, \qquad h_1 \geq h_2 \geq |h_3|\,, \tag{2}$$

where σ_i, $i = 1, 2, 3$, stand for the Pauli matrices. In the rest of the paper H denotes a Hamiltonian written in its

canonical form and $\vec{h}$ denotes the vector (h_1, h_2, h_3) with its properly ordered coefficients. The special majorization relation $\vec{x} \prec_s \vec{y}$ between three-dimensional real vectors $\vec{x}$ and $\vec{y}$ is relevant in this context. It is given by the set of inequalities

$$\begin{aligned} x_1 &\le y_1, \\ x_1 + x_2 - x_3 &\le y_1 + y_2 - y_3, \\ x_1 + x_2 + x_3 &\le y_1 + y_2 + y_3. \end{aligned} \tag{3}$$

where the components x_i and y_i are assumed to fulfill $x_1 \ge x_2 \ge |x_3|$ and $y_1 \ge y_2 \ge |y_3|$ [8].

Fact 1 (Theorem of [3]): The minimal time overhead $t_{H'|H}$ (inverse of the efficiency $s_{H'|H}$ of [3]) required to simulate Hamiltonian H' by Hamiltonian H and fast LU is the minimal value of $c \ge 0$ such that the vectors $\vec{h}'$ and $\vec{h}$ satisfy $\vec{h}' \prec_s c\vec{h}$. Protocols for optimal simulation are known.

(*ii*) *Optimal synthesis of two-qubit gates under LU.*— Any two-qubit gate U can be written in terms of local unitaries u_i and v_i and Pauli matrices σ_k as [1,5]

$$U = (u_1 \otimes v_1) e^{-i \sum_k \lambda_k \sigma_k \otimes \sigma_k} (u_2 \otimes v_2). \tag{4}$$

In [5] it is shown how to obtain this decomposition. Notice that gate U is equivalent, up to local unitaries performed on the qubits before and after U, to

$$U_{\vec{\lambda}} \equiv e^{-i \sum_k \lambda_k \sigma_k \otimes \sigma_k}. \tag{5}$$

Since we assume the ability to perform instantaneous (i.e., sufficiently fast) LU operations, the synthesis of U is as time consuming as that of $U_{\vec{\lambda}}$, and we need only focus on the latter. In addition [6], to each U there corresponds a unique $U_{\vec{\lambda}^0}$ with $\vec{\lambda}^0 \equiv (\lambda_1^0, \lambda_2^0, \lambda_3^0)$ such that $\lambda_1^0 \ge \lambda_2^0 \ge |\lambda_3^0|$, $\lambda_1^0, \lambda_2^0 \in [0, \pi/4]$, and $\lambda_3^0 \in (-\pi/4, \pi/4]$, that we shall call its canonical form. In what follows we will often represent any two-qubit gate U by its canonical form $U_{\vec{\lambda}^0}$ or by its corresponding (unique) vector $\vec{\lambda}^0 = (\lambda_1^0, \lambda_2^0, \lambda_3^0)$. Recall that all commutators $[\sigma_j \otimes \sigma_j, \sigma_k \otimes \sigma_k]$ vanish, and that $\exp(-i\pi/2 \sigma_j \otimes \sigma_j) = -i\sigma_j \otimes \sigma_j$ is a local gate. This implies that, for any vector $\vec{n} = (n_1, n_2, n_3)$ with integer components n_j,

$$U_{\vec{\lambda}^0} =_{\mathrm{LU}} U_{\vec{\lambda}^0} U_{\frac{\pi}{2}\vec{n}} = U_{\vec{\lambda}^0 + (\pi/2)\vec{n}}, \tag{6}$$

with $\vec{\lambda}^0 + \pi/2\vec{n}$ essentially exhausting all vectors compatible with the gate $U_{\vec{\lambda}^0}$ [9]. ($=_{\mathrm{LU}}$ is used to denote equivalence under LU.)

In Theorem 10 of [1] the problem of time-optimally producing a two-qubit gate U using interaction H is shown to reduce to a specific minimization over all possible decompositions of U of the form (4). Here we rephrase the theorem in terms of the notion of Hamiltonian simulation and the concepts introduced earlier. Without loss of generality, we refer only to unitary operations that can be written as in (5), and associate a self-adjoint operator $H_{\vec{\lambda}} \equiv \sum_i \lambda_i \sigma_i \otimes \sigma_i$ to each such decomposition.

Fact 2 (Theorem 10 of [1], readapted): The time-optimal way to synthesize gate U with interaction H and fast LU consists of simulating, among all Hamiltonians $H_{\vec{\lambda}}$ such that $U = \exp(-iH_{\vec{\lambda}})$, the one with smallest time overhead $t_{H_{\vec{\lambda}}|H}$. The minimal interaction time [i.e., the interaction cost $C_H(U)$] is given by the smallest time overhead $t_{H_{\vec{\lambda}}|H}$.

Our first aim is to perform the optimization described in Fact 2. This is feasible because we have an analytical characterization both of all (infinitely many) decompositions of U [cf. Eq. (6)] and of the time overhead $s_{H_{\vec{\lambda}}|H}$ for any decomposition (cf. Fact 1), as expressed in the following lemma.

Lemma: The interaction cost $C_H(U)$ is the minimal value $c \ge 0$ such that a vector $\vec{n}$ of integers exists satisfying

$$\vec{\lambda}^0 + \frac{\pi}{2}\vec{n} \prec_s c\vec{h}. \tag{7}$$

It is useful to introduce, for each $\vec{n}$, the *precost* $c_{\vec{n}}$ as the minimal value $c \ge 0$ such that $\vec{\lambda}^0 + \pi/2\vec{n} \prec_s c\vec{h}$. Precost $c_{\vec{n}}$ is the overhead needed to simulate $H_{\vec{\lambda}^0 + \pi/2\vec{n}}$ by H or, equivalently, the minimal time t needed to travel, in the set of nonlocal gates, from the identity operator to U along the path defined by $\vec{\lambda}^0 + \pi/2\vec{n}$. Intuitively, a large $\vec{n}$ corresponds to a "long"—and therefore nonoptimal—path. Following this intuition we arrive at our first result.

Theorem 1: The interaction cost $C_H(U)$ or minimal time needed to create gate U by using Hamiltonian H and fast LU is given by

$$C_H(U) = \min\{c_{(0,0,0)}, c_{(-1,0,0)}\}, \tag{8}$$

that is, the minimum of two precosts, one corresponding to the canonical vector $\vec{\lambda}^0 = (\lambda_1^0, \lambda_2^0, \lambda_3^0)$ of U and the other to the vector $(\frac{\pi}{2} - \lambda_1^0, \lambda_2^0, -\lambda_3^0)$ [10]. The time-optimal protocol consists of simulating the corresponding Hamiltonian [either $\vec{h}_1 \equiv (\lambda_1^0, \lambda_2^0, \lambda_3^0)$ or $\vec{h}_2 = (\frac{\pi}{2} - \lambda_1^0, \lambda_2^0, -\lambda_3^0)$] by H for time $t = C_H(U)$.

Remark: Thus, in order to time-optimally perform gate U with Hamiltonian H, we can proceed as follows. By using Ref. [5], we compute $\vec{\lambda}^0$ from U, and by using Refs. [3,4], we compute $\vec{h}$ from H. Theorem 1 gives the minimal time of simulation and the Hamiltonian (either $\vec{h}_1$ or $\vec{h}_2$) to be simulated, and finally Ref. [3] describes an optimal protocol for simulating the convenient Hamiltonian by H and LU.

Proof: We need to see that $C_H(U)$ as given by Eq. (8) is the minimal precost, i.e., $C_H(U) \le c_{\vec{n}}$ for all $\vec{n}$. It is straightforward to check from Eq. (3) that (T1.*i*) for any two vectors $\vec{x}$ and $\vec{y}$, with components $x_1 \ge x_2 \ge |x_3|$ and $y_1 \ge y_2 \ge |y_3|$, the minimal $c \ge 0$ such that $\vec{x} \prec_s c\vec{y}$ satisfies $c \le 3x_1/y_1$; (T1.*ii*) if $\vec{x} \prec_s \vec{x}'$, then $\vec{x}' \prec_s \vec{y} \Rightarrow \vec{x} \prec_s \vec{y}$ ($\prec_s$ is a partial order). In particular, let $c' \ge 0$ be the minimal value such that $\vec{x}' \prec_s c'\vec{y}$. Then

$\vec{x} \prec_s \vec{x}' \Rightarrow \vec{x} \prec_s c'\vec{y}$, so that the minimal $c \geq 0$ such that $\vec{x} \prec_s c\vec{y}$ always satisfies $c \leq c'$. Now, recall that, by definition $\pi/4 \geq \lambda_1^0 \geq 0$, and notice that, if some component n_j of $\vec{n}$ fulfills $|n_j| > 1$, then the maximal component of the reordered version [8] of $\vec{\lambda}^0 + \pi/2\vec{n}$ is at least $3\pi/4$. Thus, because of (T1.*i*), $\vec{\lambda}^0 \prec_s \vec{\lambda}^0 + \pi/2\vec{n}$. Then (T1.*ii*) implies that $c_{(0,0,0)} \leq c_{\vec{n}}$. Therefore we can restrict our attention to vectors $\vec{n}$ with $|n_j| \leq 1$. A case-by-case check shows that the precosts $c_{\vec{n}}$ with $\vec{n} \in \{(-1,-1,-1), (0,-1,0), (0,0,-1), (0,0,1)\}$ fulfill $c_{(-1,0,0)} \leq c_{\vec{n}}$, since (cf. point (T1.*ii*) above) $\vec{\lambda}^0 + \pi/2(-1,0,0) \prec_s \vec{\lambda}^0 + \pi/2\vec{n}$ [11]. Similarly, we obtain that for the remaining vectors $\vec{n}$ with $|n_j| \leq 1$ the precosts satisfy $c_{(0,0,0)} \leq c_{\vec{n}}$, because $\vec{\lambda}^0 + \pi/2(0,0,0) \prec_s \vec{\lambda}^0 + \pi/2\vec{n}$. The only remaining configurations, with vectors $\vec{n} \in \{(-1,0,0), (0,0,0)\}$, are incomparable according to the $\prec_s$ relation—unless $\lambda_1^0 + |\lambda_3^0| \leq \pi/4$, in which case we always obtain $c_{(0,0,0)} \leq c_{(-1,0,0)}$—, and this is why the optimization of Eq. (8) has to be performed. □

Corollary: (a) When U is such that $\lambda_1^0 + |\lambda_3^0| \leq \pi/4$, then the interaction cost is always given by

$$C_H(U) = c_{(0,0,0)}\,. \tag{9}$$

(b) If, instead, $\lambda_1^0 + |\lambda_3^0| < \pi/4$, then Hamiltonians H and H' always exist such that $C_H(U) = c_{(0,0,0)}^{\vec{h}} < c_{(-1,0,0)}^{\vec{h}}$ and $C_{H'}(U) = c_{(-1,0,0)}^{\vec{h}'} < c_{(0,0,0)}^{\vec{h}'}$.

Proof: (a) follows from the fact that $\lambda_1^0 + |\lambda_3^0| \leq \pi/4 \Rightarrow \vec{\lambda}^0 \prec_s \vec{\lambda}^0 + \pi/2(-1,0,0)$. (b) can be checked by considering H and H' given by $\vec{h} = \vec{\lambda}^0$ and $\vec{h}' = (\pi/2 - \lambda_1^0, \lambda_2^0, -\lambda_3^0)$. □

In order to analyze Eq. (8) we first consider some examples. For the Ising interaction $H = h\sigma_3 \otimes \sigma_3$ (equivalently $h\sigma_1 \otimes \sigma_1$) and an arbitrary gate U, Eq. (8) reads (cf. Theorem 2 of [1]),

$$C_{h\sigma_1\otimes\sigma_1}(U) = \frac{\lambda_1^0 + \lambda_2^0 + |\lambda_3^0|}{h}\,. \tag{10}$$

Let us now instead focus on three specific gates and arbitrary interactions. By $|m\rangle \otimes |n\rangle$ $(m, n = 0, 1)$ we denote the computational basis of two qubits. The CNOT gate is defined as

$$|m\rangle \otimes |n\rangle \rightarrow |m\rangle \otimes |n \oplus m\rangle\,, \tag{11}$$

where $\oplus$ is sum modulo 2. Using the method described in Ref. [5] we obtain its canonical vector, $\vec{\lambda}^0 = (\pi/4, 0, 0)$. Similarly, the SWAP gate,

$$|m\rangle \otimes |n\rangle \rightarrow |n\rangle \otimes |m\rangle\,, \tag{12}$$

has vector $\vec{\lambda}^0 = (\pi/4, \pi/4, \pi/4)$. We also consider a third, intermediate gate U_{XY} with $\vec{\lambda}^0 = (\pi/4, \pi/4, 0)$, that corresponds to

$$|m\rangle \otimes |n\rangle \rightarrow i^{|m-n|}|n\rangle \otimes |m\rangle\,. \tag{13}$$

For these three gates we find

$$C_H(\text{CNOT}) = \frac{\pi}{4}\frac{1}{h_1}, \tag{14}$$

$$C_H(U_{XY}) = \frac{\pi}{4}\frac{2}{h_1 + h_2 - |h_3|}\,, \tag{15}$$

$$C_H(\text{SWAP}) = \frac{\pi}{4}\frac{3}{h_1 + h_2 + |h_3|}\,. \tag{16}$$

With these examples at hand we make the following two observations. First, to any gate U there corresponds a *natural interaction* H_U, with vector either $\vec{h}_1$ or $\vec{h}_2$ as defined in Theorem 1. This natural interaction allows one to perform gate U optimally without need to simulate an intermediate another Hamiltonian, and therefore the time inefficiencies inherent in the process of simulation are avoided. The natural interactions for the CNOT gate, gate U_{XY} and the SWAP gate are, respectively, the Ising interaction $\sigma_1 \otimes \sigma_1$, the XY-model interaction $\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2$, and the Heisenberg or exchange interaction $\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3$.

The second observation is that for any fixed interaction H the interaction cost induces an order in the set of gates. For instance, according to Eq. (10), a SWAP is the most time-consuming gate when the Ising interaction is available. Equations (14)–(16) also show that such an order depends on the available interaction. Using the exchange interaction $H = \sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3$, U_{XY} is twice as time consuming as a SWAP gate.

Let us move to Definition 2. It endows the set of nonlocal gates with a partial order structure based on the notion of interaction cost, but which is independent of any particular interaction. By comparing the resources required to perform two gates, such a partial order captures the intuition that some gates are more nonlocal than others.

We have argued that no gate exists such that it is more nonlocal (i.e., more time consuming for all interactions) than all the others. It is also easy to see that a gate $\alpha\vec{\lambda}^0$ is always less nonlocal than $\vec{\lambda}^0$ for any $\alpha \in [0, 1]$, since the precosts are linear in α. Next we present an analytical characterization of the partial order relation $V \leq U$ in a region of the set of two-qubit gates [12].

Theorem 2: Let U and V be two two-qubit gates with corresponding ordered vectors $\vec{\lambda}_U^0$ and $\vec{\lambda}_V^0$ such that in both cases the restriction $\lambda_1^0 + |\lambda_3^0| \leq \pi/4$ holds. Then gate U is more nonlocal than gate V if, and only if, $\vec{\lambda}_V^0 \prec_s \vec{\lambda}_U^0$,

$$V \leq U \Leftrightarrow \vec{\lambda}_V^0 \prec_s \vec{\lambda}_U^0\,. \tag{17}$$

Proof: Recall that the restrictions on $\vec{\lambda}_U^0$ and $\vec{\lambda}_V^0$ imply, because of Corollary 1, that the interaction costs $C_H(U)$ and $C_H(V)$ are given, respectively, by the smallest $c_U, c_V \geq 0$ such that

$$\vec{\lambda}_U^0 \prec_s c_U\vec{h}\,, \tag{18}$$

$$\vec{\lambda}_V^0 \prec_s c_V\vec{h}\,. \tag{19}$$

Suppose first that $V \leq U$, that is, for any Hamiltonian H we have $C_H(V) \leq C_H(U)$. Then we also have $\vec{\lambda}_V^0 \prec_s C_H(U)\vec{h}$. In particular, if we choose the interaction H to have vector $\vec{h} = \vec{\lambda}_U^0$, we have $C_H(V) \leq C_H(U) = 1$ and $\vec{\lambda}_V^0 \prec_s C_H(U)\vec{h} = \vec{\lambda}_U^0$, which proves the direct implication. The inverse implication follows from (T1.*ii*) of the proof of Theorem 1, which shows that $\vec{\lambda}_V^0 \prec_s \vec{\lambda}_U^0$ implies that $C_H(V) \leq C_H(U)$ for all H. □

As an example of this result, the U_{XY} gate is more nonlocal than the CNOT gate and, as it was to be expected, gates with sufficiently small components $|\lambda_i^0|$ are less nonlocal than those with large $|\lambda_i^0|$ [13].

In this Letter we have characterized the time-optimal synthesis of two-qubit unitary transformations using an arbitrary two-qubit Hamiltonian. In particular, the interaction cost $C_H(H)$ has been computed and optimal protocols have been described. We have also characterized, in a region of the space of two-qubit gates, a partial order structure related to their degree of nonlocality. These results can be applied to the study of the interaction cost for particular processes, such as the creation of a maximally entangled state [4,5] or the transmition of a classical or quantum bit of information from one qubit to another [6]. All these discussions involve only two interacting qubits. It would be desirable to obtain a generalization to higher-dimensional systems. The lack of an analog to decomposition (4) in these cases is a serious drawback.

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[7] Several operational approaches can be taken to quantify the degree of nonlocality of a unitary transformation U. As in [5], for instance, one can consider the increase ΔE in entanglement that some optimal state experiences when transformed according to U. This, however, captures only some aspects of U, since its entangling power is not limited to its action on a single state. The partial order introduced in Definition 2 is constructed in relation to the interaction time required to perform U. Interaction times offer the opportunity to introduce a notion of nonlocality for operation U which is independent of a choice of state on which U acts. Accordingly, this notion need, in principle, not be quantitatively equivalent to that of nonlocality for entangled states.

[8] The s-majorization relation can be extended to non-ordered vectors as follows. Given a vector $\vec{u} = (u_1, u_2, u_3)$, we construct a new vector $\vec{u}^s = (u_1^s, u_2^s, u_3^s)$, $u_1^s \geq u_2^s \geq |u_3^s|$ by first decreasingly reordering the modulus of the components u_i, and by then giving u_3^s the sign of the product $u_1 u_2 u_3$. Then, for any pair of vectors $\vec{u}$ and $\vec{v}$, $\vec{u} \prec_s \vec{v}$ denotes the set of inequalities (3) applied to $\vec{u}^s$ and $\vec{v}^s$.

[9] By permuting the components of $\vec{\lambda}^0 + \pi/2\vec{n}$, as well as multiplying by -1 two of the components, other vectors can be built that also correspond to a decomposition of U. These are, nevertheless, equivalent to the original decomposition from the perspective of the s-majorization relation $\prec_s$, and need not be analyzed for the purposes of optimal gate synthesis.

[10] $(\frac{\pi}{2} - \lambda_1^0, \lambda_2^0, -\lambda_3^0)$ is the reordered version [8] of $\vec{\lambda}^0 + \pi/2(-1, 0, 0)$.

[11] Let us consider, as an example, $\vec{n} = (0, 0, 1)$. If we properly reorder [8] the coefficients, we obtain the vector $(\pi/2 - \lambda_1^0, \lambda_2^0, -\lambda_3^0)$ from $\vec{\lambda}^0 + \pi/2(-1, 0, 0)$ and the vector $(\pi/2 + \lambda_3^0, \lambda_1^0, \lambda_2^0)$ from $\vec{\lambda}^0 + \pi/2(0, 0, 1)$. Then

$$\pi/2 - \lambda_1^0 \leq \pi/2 + \lambda_3^0,$$

$$\pi/2 - \lambda_1^0 + \lambda_2^0 + \lambda_3^0 \leq \pi/2 + \lambda_3^0 + \lambda_1^0 - \lambda_2^0,$$

$$\pi/2 - \lambda_1^0 + \lambda_2^0 - \lambda_3^0 \leq \pi/2 + \lambda_3^0 + \lambda_1^0 + \lambda_2^0,$$

so that indeed $\vec{\lambda}^0 + \pi/2(-1, 0, 0) \prec_s \vec{\lambda}^0 + \pi/2(0, 0, 1)$ and therefore the precosts satisfy $c_{(-1,0,0)} \leq c_{(0,0,1)}$ [cf. point (T1.*ii*) in the proof of Theorem 1].

[12] Contrary to the problem of Hamiltonian simulation, where the analogous partial order has been solved completely [3], here we do not have a linear space of Hermitian operators (Hamiltonians), but the rather involved manifold $SU(4)$. In particular, in the region surrounding $\vec{\lambda}^0 = (\pi/4, \pi/4, \pi/4)$ geometrical aspects of this manifold make the characterization very difficult. We have already met a manifestation of this fact in Eq. (8), where two precosts, instead of just one, have to be considered due to periodicity properties of unitary evolutions. The analytical characterization presented here is possible precisely away from $\vec{\lambda}^0 = (\pi/4, \pi/4, \pi/4)$.

[13] Remarkably enough, the necessary and sufficient conditions for this partial order are completely analogous to those for efficient Hamiltonian simulation [3]. Such a correspondence was, in principle, only to be expected for infinitesimal gates. It also holds for finite gates due to some convexity properties explored in Theorems 9 and 10 of [1].