Viscous Fingering and the Shape of an Electronic Droplet in the Quantum Hall Regime

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We show that the semiclassical dynamics of an electronic droplet, confined in a plane in a quantizing inhomogeneous magnetic field in the regime where the electrostatic interaction is negligible, is similar to viscous (Saffman-Taylor) fingering on the interface between two fluids with different viscosities confined in a Hele-Shaw cell. Both phenomena are described by the same equations with scales differing by a factor of up to 10^{-9}. We also report the quasiclassical wave function of the droplet in an inhomogeneous magnetic field.

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An important class of pattern formation on moving fronts occurs when diffusion, rather than convection, dominates the transport. In these cases the front moves with a normal velocity proportional to the gradient of a harmonic field, a mechanism known as D'Arcy's law or Laplacian growth (for a review, see, e.g., [1]). Viscous or Saffman-Taylor fingering is one of the most studied instabilities of this type. It occurs at the interface between two incompressible fluids with different viscosities when a less viscous fluid is injected into a more viscous one in a 2D geometry (typically, the fluids are confined in a Hele-Shaw cell, a thin gap between two parallel plates) or in porous media [2]. The interface forms a pattern of growing fingers whose shape becomes complex at high flow rates (Fig. 1).

Instabilities on diffusion driven fronts occur in different settings ranging from geological to molecular scales. In this Letter, we show that similar phenomena may take place in semiconductor nanostructures. A growth of an electronic droplet in a quantum Hall (QH) regime, while increasing the number of particles in the droplet, is similar to Laplacian growth.

We first recall the Saffman-Taylor (ST) problem. In a thin cell, the local velocity of a viscous fluid is proportional to the gradient of the pressure: $\vec{v} = -\nabla p$ (D'Arcy's law). Incompressibility implies that the pressure $p(z)$ is a harmonic function of $z = x + iy$ with a sink at infinity:

$$\nabla^2 p(z) = 0, \qquad p(z) \to -\tfrac{1}{2}\log|z| \quad \text{as } |z| \to \infty. \tag{1}$$

If the difference between viscosities is large, the pressure is constant in the less viscous fluid and, if the surface tension is ignored, it is also constant (set to zero) on the interface C. Thus, on the interface,

$$p(z) = 0, \qquad v_n = -\partial_n p(z), \qquad z \in C. \tag{2}$$

At constant flow rate, the area of the less viscous fluid grows linearly with time t. We set the flow rate such that the area equals πt. The other parameters, viscosity, and the width of the cell are chosen to set the pressure equal to the velocity potential. For recent advances in the study of fingering in channel and radial geometries, see [3,4].

It is convenient to rewrite Eqs. (1) and (2) in terms of the conformal map, $w(z,t)$, of the domain occupied by the more viscous fluid to the exterior of the unit disc $|w| \geq 1$ in such a manner that the source at $z = \infty$ is mapped to infinity. In terms of the conformal map, the pressure is $p = -\frac{1}{2}\log|w(z,t)|$, and the complex velocity in the viscous fluid is $v(z) = v_x - iv_y = \frac{1}{2}\partial_z \log w(z)$. On the interface, it is proportional to the harmonic measure:

$$v_n(z,t) = \tfrac{1}{2}|w'(z,t)|, \qquad z \in C. \tag{3}$$

The complex velocity is conveniently written using the Schwarz function, $S(z)$,

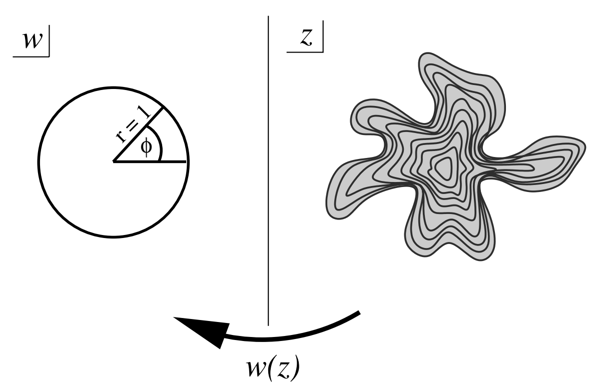

FIG. 1. A schematic illustration of a less viscous fluid domain or an electronic droplet in a strong magnetic field. The electronic droplet is stratified by semiclassical orbits. The domain of a more viscous fluid (exterior of the droplet) mapped conformally into the exterior of a unit circle.

$$\partial_z \log w(z) = \partial_t S(z).$$

The Schwarz function is analytic in a domain containing the contour C such that $S(z) = \bar{z}$ on C [5].

In idealized settings [(1) and (2)], the Saffman-Taylor problem confronts an obstacle. As a result of scale invariance, some fingers develop cusplike singularities within a finite time [6]. A modification of the growth law which introduces a mechanism curbing the curvature of the interface at a microscale is necessary.

The known cutoff mechanisms (e.g., surface tension, lattice regularization, etc.) destroy the mathematical structure of the idealized problem and make it difficult for analytical analysis. It is believed, however, that once a steady fractal pattern has been developed, its fractal character does not depend on the mechanism stabilizing the singularities.

The purpose of this Letter is twofold. One is to show that a natural mechanism which introduces a scale but captures the physical (and the mathematical [7]) structures of the problem is "quantization." Quantization implies that a change of the area of the domain is quantized. This ensures that no physical quantity can have features on a scale less than a "Planck constant" which cuts off singularities. The Saffman-Taylor problem [(1) and (2)] then arises as a semiclassical limit, when the scale introduced by Planck's constant tends to zero.

Our second goal is to show that the quantized Saffman-Taylor problem describes an electronic droplet in a special QH regime where electrostatic forces are weak. This correspondence suggests that the edge of the QH droplet may develop unstable features similar to the fingers observed in a Hele-Shaw cell (a fingering pattern is illustrated in Fig. 1). A spatial resolution of nanoscale structures in GaAs by recently developed scanning techniques might be apt for a search for such features [8].

We will study a shape of a large electronic droplet on the fully occupied lowest Landau level of a quantizing magnetic field. The magnetic field is assumed to be nonuniform in the area away from the droplet. We show that Aharonov-Bohm forces, associated with the nonuniform part of the magnetic field, shape the edge of the droplet in a manner similar to a fingering interface driven by a Laplacian field.

In order to present our argument, we shall neglect the interactions among the electrons and assume that the external electrostatic potential is zero. Under these conditions, we will show that the semiclassical dynamics of the QH droplet is governed by the same equations of viscous fingering scaled to a nanometer scale. By the semiclassical limit, we mean a large number of electrons N, small magnetic length, ℓ, but a finite area of the droplet:

$$N \to \infty, \quad \ell \equiv \sqrt{\hbar c/eB_0} \to 0, \quad t = 2\ell^2 N = \text{finite}.$$

The droplet grows when its quantized area changes by a quantum, $\pi t \to \pi t + 2\pi\ell^2$, as an electron is added ($N \to N + 1$), or by changing the magnetic length.

Prior to computations, we would like to give an insight as to why an electronic droplet in a magnetic field may behave similarly to viscous fingering. Equations (1) and (2) imply that the harmonic moments of the viscous fluid domain,

$$t_k = -\frac{1}{\pi k}\int_{\text{visc.fluid}} z^{-k}\, d^2z, \qquad k = 1, 2, \ldots .$$

do not change in time [9]. They are initial data of the evolution. Indeed, $\frac{d}{dt} t_k = \frac{1}{\pi k}\oint_C z^{-k}\partial_n p(z)\,|dz| = \frac{1}{2\pi i k}\oint_C z^{-k}\partial_z \log w(z)\, dz = 0$ since $\partial_z \log w(z)$ is analytic in the viscous domain. Conservation of the harmonic moments is an equivalent formulation of D'Arcy's law (2).

As shown below, the harmonic moments of the QH droplet do not depend on the number of particles in the droplet. The moments feature the inhomogeneity of the magnetic field. A proof of this assertion would yield D'Arcy's law for the QH droplet.

Our proof consists of two steps. First, we construct the wave functions of the QH states in a weakly nonuniform magnetic field, and introduce the τ function, a generating function of these wave functions. Then we determine semiclassical states, the orbits (supports of the wave functions), and study the evolution of the orbit as the area of the droplet is increased. We will consider droplets with smooth shape, leaving analysis of the finger-tip singularities for further publications. The size of the Letter forces us to rely on some formulas on the evolution of conformal maps proven in [10].

Let us first recall the concept of a QH droplet (see, e.g., [11,12]). Consider spin polarized planar electrons, with g-factor equaling two, in the lowest level of a quantizing nonuniform magnetic field, directed perpendicular to the plane, $B(x, y) > 0$:

$$H = \frac{1}{2m}[(-i\hbar\vec{\nabla} - \vec{A})^2 - \hbar B].$$

The lowest level of the Pauli Hamiltonian is degenerate even for a nonuniform field. The degeneracy equals the integer part of the total magnetic flux $\Phi = \int dx\, dy\, B$ in units of flux quanta, $\Phi_0 = 2\pi\hbar$ (we set $e = c = 1$). The orthonormalized degenerate states, written in transversal gauge, are $\psi_n(x, y) = P_n(z)\exp\frac{W}{\hbar}$, where the potential $W(x, y)$ obeys the equation $B = -\nabla^2 W$, and $P_n(z)$ are holomorphic polynomials of a degree n, which does not exceed the degeneracy of the level [13].

We will consider the following arrangement: A strong uniform magnetic field $B_0 > 0$ is situated in a large disk of radius R_0; the disk is surrounded by a large annulus $R_0 < |z| < R_1$ with a magnetic field $B_1 < 0$ directed opposite to B_0, such that the total magnetic flux Φ of the disk, $|z| < R_1$, is $N\Phi_0$. The magnetic field outside the disk $|z| < R_1$ vanishes. The disk is connected through a tunnel barrier to a large capacitor that maintains a uniform small positive chemical potential slightly above the zero energy of the lowest Landau level.

In this arrangement, a circular droplet of N electrons is trapped at the center of the disk $|z| < R_0$. We choose the magnetic field B_1 such that its radius $\ell\sqrt{2N}$ is much smaller than the radius of the disk R_0.

Next we assume that a weakly nonuniform magnetic field δB is placed inside the disk $|z| < R_0$ but well away from the droplet. The nonuniform magnetic field does not change the total flux $\int \delta B\, dx\, dy = 0$. The droplet grows when B_1 is adiabatically increased, keeping B_0 and δB fixed. Thus, the degeneracy of the Landau level is enlarged, and consequently, an electron enters the system.

The wave function of the droplet is the Jastrow function:

$$\Psi(z_1, \ldots, z_N) = \frac{1}{\sqrt{N! \tau_N}} \Delta(z) e^{1/\hbar \sum_n W(z_n)},$$

where

$$\Delta(z) = \prod_{n<m<N} (z_n - z_m) = \sqrt{\tau_N} \det[P_n(z_m)]|_{n,m<N}$$

is the Vandermonde determinant, and the normalization factor, the τ function,

$$\tau_N = \frac{1}{N!} \int |\Delta(\xi)|^2 \prod_n e^{(2/\hbar)W(\xi_n)}\, d^2\xi_n\,. \qquad (5)$$

depends on the magnetic field.

In terms of the orthonormal one-particle states, $\psi_n(z)$, the density $\rho_N(z) = \int |\Psi(z, \xi_1, \ldots, \xi_{N-1})|^2 d^2\xi$ reads

$$\rho_{N+1}(z) - \rho_N(z) = |\psi_N(z)|^2\,.$$

This formula defines a growth process. The one-particle state gives the probability of adding an extra particle at point z of the droplet.

The integral in (5) converges in the area of the droplet, where the magnetic field is uniform. In this area the annulus, $R_0 < |z| < R_1$, does not contribute to the potential W, and the potential V of the nonuniform part of the magnetic field $\delta B = -\nabla^2 V$ is a harmonic function:

$$V(z) = \mathrm{Re} \sum_{k\geq 1} t_k z^k, \qquad t_k = \frac{1}{2\pi k} \int \delta B(z) z^{-k}\, d^2 z\,.$$

The parameters t_k are, now, the harmonic moments of the deformed part of the magnetic field. Summing up, we have $W(z) = -\hbar|z|^2/4\ell^2 + V(z)$.

The one-particle state $\psi_N(z)$ can be expressed in terms of the τ function (later below we set $2\ell^2 = \hbar$):

$$\psi_N(z) = \frac{e^{-(1/\hbar)[(|z|^2/2) - V(z)]}}{\sqrt{\tau_{N+1}\tau_N}} z^N e^{-\hbar D(z)} \tau_N\,, \qquad (6)$$

where $D(z) = \sum_{k\geq 1} \frac{z^{-k}}{k} \frac{\partial}{\partial t_k}$ is a generator of the deformations of the magnetic field.

The next step is a semiclassical analysis of the wave function, $\psi_N(z)$. In this limit, the droplet has a sharp boundary (we assume that the boundary is smooth), and the semiclassical states (with large N) are localized at orbits of width of the order of ℓ. Since the magnetic field in the area of the droplet is assumed to be uniform, the density is also uniform inside the droplet. The area of the droplet is quantized as $\pi t = 2\pi N \ell^2$.

As $\hbar \to 0$ with $t = \hbar N$ fixed, the integral (5) is saturated by its saddle point. At the saddle point, the ξ_n are uniformly distributed in a domain determined by the area $\pi\hbar N$ and the harmonic moments t_k [10]. We will see that this domain determines the shape of the droplet. At the saddle point, the τ function (5) tends to the classical τ-function $F = \lim_{\hbar\to 0} \hbar^2 \log \tau_N$, studied in [10]:

$$F(t; t_1, \ldots) = -\frac{1}{\pi^2} \iint_{\mathrm{droplet}} \log \left| \frac{1}{z} - \frac{1}{z'} \right| d^2 z\, d^2 z'.$$

On expanding (6) in $\hbar$, and defining

$$\Omega(z) = \sum_{k\leq 1} t_k z^k + t \log z - \left(\frac{1}{2} \partial_t + D(z) \right) F\,,$$

and

$$\mathcal{A}(z, \bar{z}) = \tfrac{1}{2} |z|^2 - \mathrm{Re}\Omega(z)\,,$$

one can write the amplitude of a semiclassical state, in terms of variations of the classical τ function:

$$|\psi_N(z)|^2 \simeq \frac{e^{-(1/2)[\partial_t^2 - 2\,\mathrm{Re} D^2(z)]F}}{\sqrt{2\pi^3 \hbar}} e^{-(2/\hbar)\mathcal{A}(z,\bar{z})}, \quad t = \hbar N\,. \qquad (7)$$

Properties of the functions $\Omega(z)$ and $D^2(z)F$ clarify the meaning of a semiclassical state. We list them below. The function $\mathcal{A}(z)$ is the imaginary part of the semiclassical action. The following has been proven in [4,10].

(i) $\mathcal{A}$ reaches its minimum on a closed contour (an orbit) which bounds the domain with harmonic moments t_k and the area πt, and everywhere on the orbit the action vanishes:

$$\mathcal{A}(z) = 0, \quad \mathrm{Re}\Omega(z) = \tfrac{1}{2}|z|^2, \quad z \in \text{orbit}.$$

(ii) The first derivative of the action normal to this contour also vanishes on the orbit:

$$\partial_z \mathcal{A}(z) = 0, \quad \partial_z \Omega(z) = \bar{z}, \quad z \in \text{orbit}. \qquad (8)$$

(iii) If the area of the contour increases, while t_k remain fixed, the action changes as

$$\partial_t \mathcal{A} = -\partial_t \mathrm{Re}\Omega(z, t) = -\log|w(z, t)|\,.$$

Equation (8) implies that $S(z) \equiv \partial_z \Omega(z)$ is the Schwarz function of the contour.

The action $\mathcal{A}$ remains positive in the vicinity of the contour and everywhere in the exterior domain. Its second variation normal to the contour reads

$$\mathcal{A}(z + \delta_n z) = |\delta_n z|^2 + \ldots, \qquad z \in C\,,$$

where $\delta_n z$ is a normal deviation from a point z on the contour [we use the relation $|\delta_n z|^2 = -S'(z)(\delta_n z)^2$].

A semiclassical state is localized at the minimum of $\mathcal{A}(z)$, where the amplitude has a sharp maximum. A contour where the action vanishes is the orbit. We conclude that all orbits have the same harmonic moments t_k and differ by the area. This implies the D'Arcy law.

The second variation of the classical τ function was also computed in Ref. [10]:

$$\log \frac{w(z) - w(z')}{z - z'} = D(z)D(z')F - \frac{1}{2}\partial_t^2 F.$$

Merging the points z and z', we find that the value of the first factor in (7) is just the harmonic measure $|w'(z)|$.

Summing up, the amplitude of the semiclassical wave function in a weakly inhomogeneous magnetic field is

$$|\psi_N(z)|^2 \simeq \frac{|w'(z)|}{\pi\sqrt{2\pi\hbar}} e^{-(2/\hbar)\mathcal{A}(z,\bar{z})}, \qquad t = N\hbar.$$

The factor in front of the exponent ensures the correct normalization: $|\psi_N|^2 ds = d\phi(2\pi^3\hbar)^{-1/2}e^{-(2/\hbar)|\delta_n z|^2}$ (here ds is an element of the length of the orbit, and $d\phi$ is an element of the angle on the unit circle). The classical probability distribution is then $|\psi_N|^2 \simeq \frac{1}{2\pi}|w'(z)|\delta_C(z)$, where the δ function is localized on the orbit.

When a new particle is added to the droplet, the edge advances. The velocity v_n of the advance is defined as $\partial_t\rho(z) = v_n\partial_n\rho(z)$. The normal derivative of the density is $\partial_n\rho = \frac{1}{\pi\hbar}\delta_C(z)$, and a change of the density is $\hbar\partial_t\rho(z) \simeq \rho_{N+1} - \rho_N = |\psi_N|^2 \simeq \frac{1}{2\pi}|w'(z)|\delta_C(z)$. This prompts D'Arcy's law (3).

It is interesting to compare dimensionless viscosity of liquids used in viscous fingering experiments [14], and the "viscous" effect of quantum interference at the first Landau level. The parameter of the dimension of length controlling viscous fingers in fluids—an effective capillary number—is $\frac{2\pi}{q}\frac{b^2}{12\eta}\sigma$, where q is the flow rate, b is the thickness of the cell, η is the viscosity, and σ is the surface tension. In recent experiments [14], this number stays in the high hundreds of nanometers, but can be easily decreased by increasing the flow rate. This length is to be compared with the magnetic length ℓ. At magnetic field about 2 T it is about 50 nm. At these numbers, the capillary effects are less important in electronic liquids than in classical fluids, but remarkably stay in the same range. Under conditions reducing electrostatic effects, fingering in electronic droplets may be even more dramatic than in viscous fluids. Semiconductor devices imitating a channel geometry of the original Saffman-Taylor experiment [2] may facilitate fingering instability.

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