Tip Splittings and Phase Transitions in the Dielectric Breakdown Model: Mapping to the Diffusion-Limited Aggregation Model

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We show that the fractal growth described by the dielectric breakdown model exhibits a phase transition in the multifractal spectrum of the growth measure. The transition takes place because the tip splitting of branches forms a fixed angle. This angle is η dependent but it can be rescaled onto an "effectively" universal angle of the diffusion-limited aggregation branching process. We derive an analytic rescaling relation which is in agreement with numerical simulations. The dimension of the clusters decreases linearly with the angle and the growth becomes non-ractal at an angle close to 74° (which corresponds to $\eta = 4.0 \pm 0.3$).

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Fractal growth and patterns are common phenomena of nature. The prototype model for a mathematical description of fractal growth is the diffusion-limited aggregation (DLA) model [1]. The growth in this model is determined by the electric field (called the harmonic measure) around the emerging fractal cluster. The harmonic measure possesses multifractal scaling properties and, recently, a new insight into the behavior of this measure was presented by applying the method of convoluted conformal mappings [2]. In particular, it has been demonstrated that there exists a singular behavior in the multifractal spectrum, and this singularity is a signature of a phase transition in the thermodynamics formalism of the spectrum [3]. The phase transition in the DLA cluster occurs at a specific moment q_c of probabilities of the growth. It has further been demonstrated that the geometrical property of the DLA cluster, which gives rise to this transition, is the existence of a specific branching angle for each new offspring on the cluster [4]. For DLA, this critical branching angle was found to be around 27°. Another approach, which defines a characteristic angle in the DLA process, is to consider the stability of a finger growing in a wedge [5].

The DLA model has been nicely generalized to the dielectric breakdown model (DBM) [6] where the growth probabilities at a specific site of the cluster are determined by the value of the electric field (or the harmonic measure) raised to a power η; i.e., the growth measure at the interface follows $\rho_\eta(s)\,ds \sim |E(s)|^\eta ds$, where DLA corresponds to the case $\eta = 1$. With varying values of η clusters of different geometry are grown each with their own characteristic properties of the multifractal spectrum. These multifractal properties have not been outlined before, and it is the purpose of this Letter to examine in detail possible critical points and phase transitions in the thermodynamic formalism of the harmonic measure for the dielectric breakdown model, at varying values of η.

The structure of the clusters of the DBM model emerges from an on-going proliferation and screening, hence stagnation, of branches. In particular, the protruding branches will create fjords where the harmonic measure decreases rapidly compared to what happens around the tips. The multifractal properties are best studied using the recently proposed model of iterated conformal maps [2], since the deep fjords are numerically "invisible" to the original approach where random walkers are used as probes (see [3]). The model is based on compositions of simple conformal maps $\phi_{\lambda,\theta}$ which take the exterior of the unit circle to its exterior, except for a little bump at $e^{i\theta}$ of linear size proportional to $\sqrt{\lambda}$. We shall here use the mapping introduced in [2] which produces two square root singularities which we refer to as the branch cuts, and the tip of the bump which we refer to as the microtip. The composition of these mappings is analog to the aggregation of random walkers in the off-lattice DLA model. The dynamics is given by

$$\Phi^{(n)}(w) = \Phi^{(n-1)}[\phi_{\lambda_n,\theta_n}(w)], \tag{1}$$

where $\Phi^{(n)}$ brings the exterior of the unit circle to the exterior of the aggregate of n particles. The size of the n'th bump is controlled by the parameter λ_n, and in order to achieve particles of fixed size we have that, to leading order,

$$\lambda_n = \frac{\lambda_0}{|\Phi^{(n-1)\prime}(e^{i\theta_n})|^2}. \tag{2}$$

The growth probability $\rho_1(s)$ at the interface of a DLA cluster of size n is, in the electrostatic picture, proportional to the electric field

$$\rho_1(s)\,ds \sim |E(s)|\,ds \sim \frac{ds}{|\Phi'|}. \tag{3}$$

In this case the measure on the unit circle is uniform. When the electric field in the dielectric breakdown model [6] is raised to the power η, the measure is no longer uniform,

$$\rho_\eta(\theta)d\theta \sim \rho_\eta[s(\theta)]\left|\frac{ds}{d\theta}\right|d\theta \sim \frac{|E|^\eta}{|E|}\,d\theta \sim |\Phi'(e^{i\theta})|^{1-\eta}\,d\theta\,. \tag{4}$$

Numerically, we use the Monte Carlo technique introduced

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in [7] in order to choose θ according to the distribution ρ_η. We vary the number of iterations T linearly such that for $\eta = 1.25$, $T = 50$, and for $\eta = 4$, $T = 400$.

Below, we consider both the growth measure and the harmonic measure. The growth measure is used to derive the multifractal properties, whereas the harmonic measure is used to determine the physical properties. Let us emphasize that to describe the growth we always use the growth measure. First, we consider the growth measure and the phase transition in the corresponding multifractal spectrum.

The moments of the growth probability scale with characteristic exponents, the generalized dimensions [8],

$$\int \rho^q(s)\,ds \sim (1/R)^{(q-1)D_q} \sim n^{(1-q)D_q/D}, \tag{5}$$

where n is the number of particles and R is the linear size of the cluster.

Numerically, we approximate the integral on the left-hand side by the sum of the field evaluated along the microtips of the bumps produced by the bump mappings. The field in DLA will for clusters of size 20 000 assume values below 10^{-20}, and it follows from (3) that it is impossible with the numerical precision on the unit circle ($\Delta\theta \simeq 10^{-16}$) to maintain a reasonable resolution in the physical space $\Delta s = \frac{\Delta\theta}{|E|} \simeq 10^4$. We therefore use the resolution increasing approach introduced in [3] where one keeps track on the dynamics of the branch cuts.

An easy way to see the existence of a phase transition in the multifractal spectrum is to look at the distribution of ρ_η sampled along the tips of the bumps. The distribution will for the smallest values of $\rho_\eta < c$ (below some cutoff value c) scale with a characteristic exponent $(1-\beta)/\beta$. The value of β is calculated by reordering the N computed values of ρ_η in ascending order. In other words, we write them as a sequence $\{\rho_\eta(i)\}_{i\in I}$, where I is an ordering of the indices such that $\rho_\eta(i) \le \rho_\eta(j)$ if $i < j$. We treat the discrete index i/N as a continuous index $0 \le x \le 1$ and therefore ρ_η as a nondecreasing function of x,

$$\rho_\eta \equiv f(x). \tag{6}$$

Numerically, we find that the function $f(x)$ obeys a power law with an exponent β for values of $x \ll 1$. From $f(x)$ we calculate the distribution of $p(\rho_\eta)$ by the transformation formula,

$$\begin{aligned} p(\rho_\eta) &\sim \int \delta[\rho_\eta - f(x)]\,dx \\ &= \frac{1}{|f'[x(\rho_\eta)]|} \sim \rho_\eta^{(1-\beta)/\beta}. \end{aligned} \tag{7}$$

Applying this distribution, the integral of moments (5) is rewritten as

$$\begin{aligned} \int_0^L \rho_\eta^q\,ds &= \int_0^\infty \rho_\eta^q p(\rho_\eta)\,d\rho_\eta \\ &= k\int_0^c \rho_\eta^q \rho_\eta^{(1-\beta)/\beta}\,d\rho_\eta \\ &\quad + \int_c^\infty \rho_\eta^q p(\rho_\eta)\,d\rho_\eta, \end{aligned} \tag{8}$$

where k is some normalization constant. The left integral in the final expression diverges whenever

$$q \le q_c = -\frac{1}{\beta}. \tag{9}$$

The phase transition in the multifractal spectrum of the growth measure takes place for a specific value $q = q_c$. We shall now argue that the value of β, and therefore q_c, is independent of the value of η. In other words, the phase transition in the multifractal spectrum of the growth measure takes place at a universal q_c. Figure 2a (below) shows our numerical results for the variation of q_c with η.

In [4] it was shown that the angle defining the branch splitting deep inside the fjords of DLA was given by a characteristic angle $\gamma_c(1) = \gamma_c(\eta = 1)$. This characteristic angle was also identified as the reason for the phase transition in the multifractal spectrum of the harmonic measure of DLA. The electric field along the branches in a wedge with opening angle γ scales as

$$\rho_1(x) \propto |E(x)| \sim x^{\pi/\gamma - 1}, \tag{10}$$

and thus, in the DBM model, the growth probability inside this wedge is given by

$$\rho_\eta(x) \propto |E(x)|^\eta \sim x^{\eta(\pi/\gamma - 1)}. \tag{11}$$

Therefore, when introducing the exponent η, we see that the scaling in x of the growth probability (11) inside a wedge of opening angle γ effectively becomes similar to the scaling of the electric field inside a wedge with a smaller opening angle.

As seen in Fig. 1, the topological wedge structure at the bottom of the fjords of DLA is not affected by a change in the value of η, only the opening angle is changed. Generally, the angle is changed such that $\gamma_c(\eta > 1) > \gamma_c(1)$ and $\gamma_c(\eta < 1) < \gamma_c(1)$. We now conjecture that the characteristic angle formed inside the fjords of a growing cluster exactly compensates for the change in the scaling of the growth probability when we vary η. In other words, the scaling of the growth probability inside the fjords is unaffected on the expense of a change in angle. Therefore we can derive an expression for the characteristic angle as a function of η by equating the exponents in (10) and (11):

$$\left(\frac{\pi}{\gamma_c(\eta)} - 1\right)\eta = \frac{\pi}{\gamma_c(1)} - 1, \tag{12}$$

or

$$\gamma_c(\eta) = \frac{\pi\eta}{\frac{\pi}{\gamma_c(1)} - 1 + \eta}. \tag{13}$$

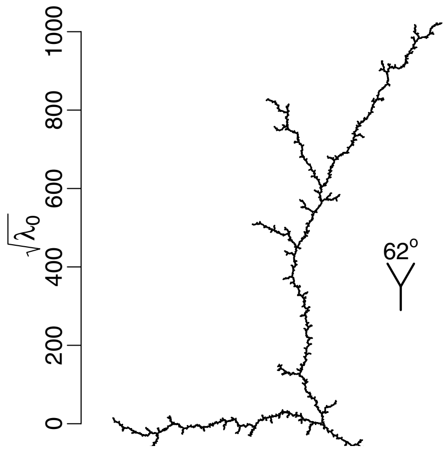

FIG. 1. Part of a cluster grown with $\eta = 3$. The wedge structure at the bottom of the fjords is clearly seen, and the opening angle observed along the aggregate is close to the angle predicted in (13) and shown in the figure.

Figure 2b shows together with numerical predictions (see below) how the critical angle varies with η. The value of $\gamma_c(1)$ in (13), and used in the figure, is determined numerically from 15 DLA clusters of size $n = 20\,000$, $\gamma_c(1) = 27° \pm 3°$.

The distribution of ρ_η inside a wedge with opening angle γ follows from (11), with $\alpha = \pi/\gamma$,

$$p(\rho_\eta) \sim \frac{1}{[x(\rho_\eta)]^{\eta(\alpha-1)-1}} \sim \rho_\eta^{[1-\eta(\alpha-1)]/[\eta(\alpha-1)]}, \quad (14)$$

and, if we compare the exponent with that of (7) and insert the critical angle from (12), one obtains

$$\beta = \frac{\pi}{\gamma_c(1)} - 1\,. \quad (15)$$

Therefore, under the above assumptions, the value of β in expression (9) is independent of η.

One way to numerically calculate the critical angles shown in Fig. 2b is first to locate the regions where the distribution in (7) scales and afterwards perform a direct measurement in these regions. Such measurements are most likely rather inaccurate and therefore we turn to Eq. (10). During the growth, we apply the exponent of η as usual, but once a cluster is grown we consider the harmonic measure ρ_1 only. Similarly, as above, we find that inside a wedge

$$p(\rho_1) \sim \rho_1^{[1-(\alpha-1)]/(\alpha-1)}. \quad (16)$$

The exponent is compared to the one we obtain numerically from (7), and from this we find the critical angle as a function of β. Note that, when we consider the harmonic measure and not the growth measure, β is a nontrivial function of η and the critical angle is given by

$$\gamma_c(\eta) = \frac{\pi}{1 + \beta(\eta)}\,. \quad (17)$$

Another interesting observation is that the dimension seems to depend linearly on the critical angle as shown in Fig. 3. Note that the linear fit predicts a dimension equal to one before the angle reaches 180°, and therefore the growth is only fractal for η below a finite value η_c.

The intersection point corresponds to an angle $\gamma_c = 74°$, and using (13) we find that

$$\eta_c = 4.0 \pm 0.3\,, \quad (18)$$

in agreement with the results obtained in [7,9]. In [10] the value of the critical angle was found to be 72° by requiring that the growth occurs along the direction of the field. To put these results into perspective, we rewrite the relation between the dimension and the critical angle in terms of the angles γ_2 and γ_1 at which the growth becomes two- and one-dimensional, respectively,

$$D(\gamma_c) = 1 + \frac{\gamma_1 - \gamma_c}{\gamma_1 - \gamma_2}, \qquad \gamma_2 \le \gamma_c \le \gamma_1\,. \quad (19)$$

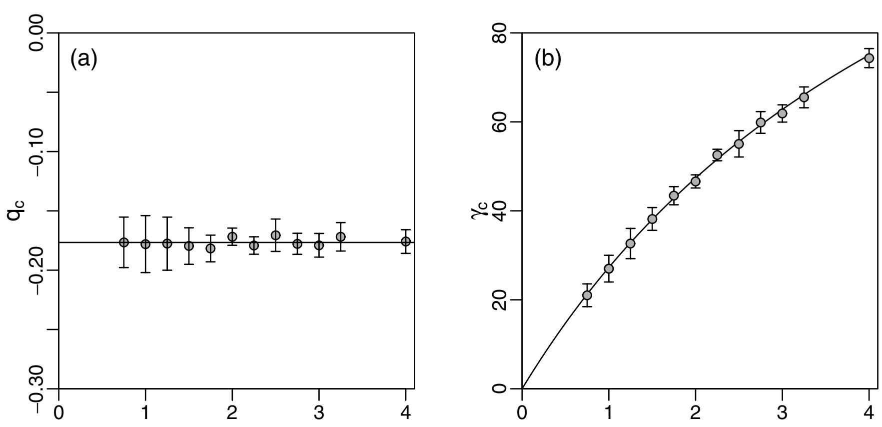

FIG. 2. (a) The critical value of the moment for the phase transition point as function of η in the multifractal spectrum of the growth measure. All the values of q_c are close to -0.18. (b) The critical angle as function of η. The line represents the critical angle obtained from the analytical expression (13). Each dot in the above figures corresponds to numerical averages of 4–15 clusters of size $n = 18\,000$.

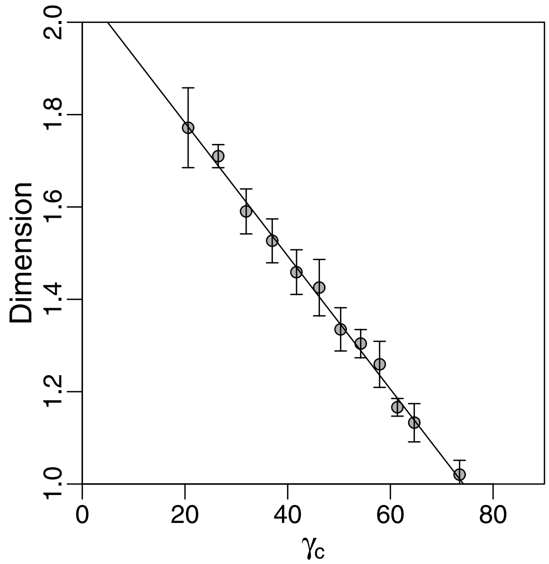

FIG. 3. The estimated dimension plotted versus the critical angle γ_c, see (13). The range of the critical angle corresponds to values of η between 0.75 and 3.5. The linear fit of the points intersects the line $D = 1$ at an angle $\gamma_c \approx 74°$.

The values of the dimension are thus determined entirely by local properties (i.e., the critical wedge angle). Other studies have also focused on local properties such as theory of branched growth in [11]. The dimension can also by (13) be written in terms of η, and in this case the dependence will no longer be linear but be on a form similar to that observed in [9]. Because of the finite size of the bumps used in the growth, we do not observe, in the limit $\eta \rightarrow 0^+$, that the critical angle of the fjords vanishes. The bumps will fill up the fjords, and the growth becomes two-dimensional for a nonvanishing value of η and $\gamma_2 \approx 5°$. The fill up problem is also the reason why we have not been able to present data points for values of η below 0.75.

In conclusion, we have studied the critical properties of the growth of clusters in the dielectric breakdown model. In particular, we have focused on the branching process and have found that the branching, on the average, occurs at a fixed angle which depends on the value of the characteristic parameter η of the DBM model. The size of the angle in turn determines the phase transition point of the growth measure. We have derived an analytic expression for the variation of the branching angle with η and found excellent agreement with numerical data. Further, we have found a linear dependence of the dimension of the cluster with the value of critical angle. This linear dependence results in a prediction of the branching angle at the point where the growth becomes one-dimensional. It is found to be $\gamma_c \approx 74^o$ corresponding to $\eta_c \approx 4$ in agreement with a result obtained in [7,9,10].

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