Quadratic Bell Inequalities as Tests for Multipartite Entanglement

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This Letter presents quantum mechanical inequalities which distinguish, for systems of n spin-$\frac{1}{2}$ particles ($n > 2$), between fully entangled states and states in which at most $n - 1$ particles are entangled. These inequalities are stronger than those obtained by Gisin and Bechmann-Pasquinucci [Phys. Lett. A **246**, 1 (1998)] and by Seevinck and Svetlichny [quant-ph/0201046].

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The Bell inequality was originally designed to test the predictions of quantum mechanics against those of a local hidden-variables theory. However, this inequality also provides a test to distinguish entangled from nonentangled quantum states. Indeed, it is well known that any nonentangled two-particle state obeys the Bell inequalities and that all pure entangled two-particle states violate them for some choice of observables [1].

With the current experimental effort to produce entangled states of three [2] and four [3] particles, it is natural to pursue n-particle generalizations of the Bell inequality that may likewise distinguish genuine multipartite entanglement from lesser entangled states. The goal of this paper is to report such inequalities for spin-$\frac{1}{2}$ particles which are stronger than previous results [4–6].

The inequalities derived here are quadratic: they employ squares of the expectation values of certain combinations of operators. Curiously, they provide tests for entanglement for systems of only three particles or more, and not for $n = 2$. At the end of this Letter, a comment is made on the reason why the present inequalities do not apply to test so-called partially separable hidden-variables theories, as considered by Svetlichny [4] and Seevinck and Svetlichny [6].

As a warming-up exercise, consider the familiar case of two spin-$\frac{1}{2}$ particles. Let A, A' denote spin observables on the first particle, and B, B' on the second. We write AB, etc., as shorthand for $A \otimes B$ and $\langle AB \rangle_\rho := \mathrm{Tr}\rho A \otimes B$; $\langle AB \rangle_\psi = \langle \psi | A \otimes B | \psi \rangle$ for the expectations of AB in the mixed state ρ or pure state $|\psi\rangle$.

The Bell inequality says that for nonentangled states, i.e., for states of the form $\rho = \rho_1 \otimes \rho_2$, or mixtures of such states,

$$|\langle AB + AB' + A'B - A'B' \rangle_\rho| \leq 2 . \tag{1}$$

The maximal violation of (1) for entangled states follows from an inequality of Cirel'son [7] (cf. Landau [8]):

$$|\langle AB + AB' + A'B - A'B' \rangle_\rho| \leq 2\sqrt{2} . \tag{2}$$

Equality in (2) can be attained by the singlet state.

The first result of this paper, and the stepping stone to the multiparticle generalizations discussed below, is that for all states ρ

$$\langle AB' + A'B \rangle_\rho^2 + \langle AB - A'B' \rangle_\rho^2 \leq 4 , \tag{3}$$

which strengthens the Cirel'son inequality (2). (A proof is given in the appendix.) Note, however, that no smaller bound on the left-hand side of (3) exists for nonentangled states. (To verify this, take $|\psi\rangle = |\uparrow\uparrow\rangle$ and $A = A' = B = B' = \sigma_z$.) Thus, the quadratic inequality (3) does not distinguish entangled and nonentangled states. But we shall see below that this is different for multiparticle generalizations of (3).

Now, consider a system of three spin-$\frac{1}{2}$ particles. In this case, we wish to distinguish between, on the one hand, states that are at most two-partite entangled, i.e., states of the form $\rho_1 \otimes \rho_{23}$, $\rho_2 \otimes \rho_{13}$, and $\rho_{12} \otimes \rho_3$, or mixtures of these states, and, on the other hand, states which are not of this form, and are called fully entangled. An example of a fully entangled state is the so-called Greenberger-Zeilinger-Horne (GHZ) state $\frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle \pm |\downarrow\downarrow\downarrow\rangle)$. Generalizations of Bell inequalities for this purpose have been presented by Svetlichny [4] and by Gisin and Bechmann-Pasquinucci [5].

As before, let A, A', B, B', and C, C' be spin observables on each of the three particles, respectively. Denote the set of all three-particle states as S_3 and the subset of states which are at most two-partite entangled as S_3^2. Svetlichny [4] obtained the following inequalities:

$$\forall\ \rho \in S_3^2: \qquad |\langle S_3^\pm \rangle_\rho| \leq 4 , \tag{4}$$

where

$$\begin{aligned} S_3^- := {} & ABC + ABC' + AB'C + A'BC - AB'C' \\ & - A'BC' - A'B'C - A'B'C', \end{aligned} \tag{5}$$

$$\begin{aligned} S_3^+ := {} & ABC - ABC' - AB'C - A'BC - AB'C' \\ & - A'BC' - A'B'C + A'B'C'. \end{aligned} \tag{6}$$

Reference [4] also showed that a pure state, unitarily equivalent to the GHZ state, yields $\langle S_3^\pm \rangle = 4\sqrt{2}$ for appropriate choices of observables. More recently, Seevinck and Svetlichny [6] show that this value is, in fact, the maximum for all three-particle states, i.e.,

$$\forall\ \rho \in S_3: \qquad |\langle S_3^\pm \rangle_\rho| \leq 4\sqrt{2} . \tag{7}$$

Gisin and Bechmann-Pasquinucci [5] obtained another inequality by means of a recursive argument from the so-called Bell-Klyshko inequality [9]. Specialized to the

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case of three particles, their results are

$$\forall\ \rho \in S_3^2: \qquad |\langle F_3\rangle_\rho| \le 2\sqrt{2}, \tag{8}$$

where

$$F_3 := ABC' + AB'C + A'BC - A'B'C', \tag{9}$$

whereas

$$\forall\ \rho \in S_3: \qquad |\langle F_3\rangle_\rho| \le 4. \tag{10}$$

Again, equality in (10) is attained for a GHZ state and appropriate observables.

Thus, both (4) and (8) provide tests to distinguish two-partite entangled states from fully entangled states in the sense that a violation of either of these inequalities is a sufficient condition for full entanglement. In order to compare the strength of both inequalities, it is useful to note that

$$S_3^\pm = \mp F_3 - F_3', \tag{11}$$

where F_3' denotes the same sum of operators as F_3, but with all primed and unprimed observables interchanged. Hence, the inequalities (4) can be rewritten as

$$|\langle F_3 \pm F_3'\rangle_\rho| \le 4. \tag{12}$$

On the other hand, since (8) holds for all choices of the observables, one can write this inequality equivalently as

$$\max|\langle F_3\rangle_\rho|, |\langle F_3'\rangle_\rho| \le 2\sqrt{2}. \tag{13}$$

It is then clear (see Fig. 1) that the inequalities (12) and (13) are independent. In particular, Eq. (13) allows the (hypothetical) case $\langle F_3\rangle = \langle F_3'\rangle = 2\sqrt{2}$, which violates (12), and similarly the (equally hypothetical) case $\langle F_3\rangle = 4$, $\langle F_3'\rangle = 0$ is allowed by (12), but forbidden by (13).

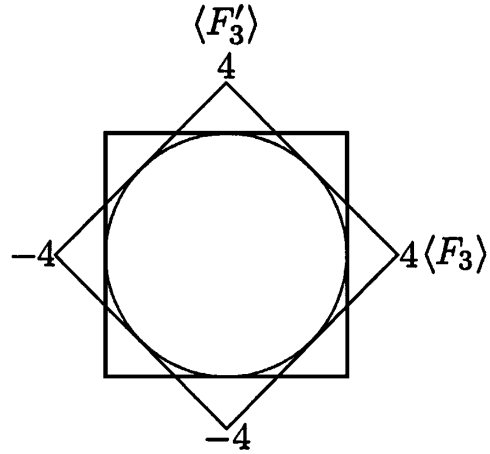

FIG. 1. Comparing the regions in the $(\langle F_3\rangle, \langle F_3'\rangle)$ plane allowed by the inequalities (13) (horizontal square), (12) (tilted square), and (15) (circle with radius $2\sqrt{2}$).

However, there exists a quadratic inequality that strengthens both (12) and (13). In fact,

$$\forall\ \rho \in S_3^2: \qquad \langle S_3^+\rangle_\rho^2 + \langle S_3^-\rangle_\rho^2 \le 16, \tag{14}$$

or equivalently, in view of (11),

$$\forall\ \rho \in S_3^2: \qquad \langle F_3\rangle_\rho^2 + \langle F_3'\rangle_\rho^2 \le 8. \tag{15}$$

Proof of (15).— Assume, for the moment, that the state is of the form

$$\rho = \rho_{12} \otimes \rho_3. \tag{16}$$

In that case, the expectations for particle 3 factorize from those for the other particles, to yield

$$\langle F_3\rangle^2 + \langle F_3'\rangle^2 = (\langle X\rangle\langle C\rangle + \langle Y\rangle\langle C'\rangle)^2 + (\langle X\rangle\langle C'\rangle - \langle Y\rangle\langle C\rangle)^2 = (\langle X\rangle^2 + \langle Y\rangle^2)(\langle C\rangle^2 + \langle C'\rangle^2) \le 8, \tag{17}$$

where I have abbreviated

$$X := AB' + A'B, \qquad Y := AB - A'B' \tag{18}$$

and used $\langle C\rangle^2 + \langle C'\rangle^2 \le 2$, and inequality (3). The proof is completed by noting that the left-hand side of (15) is invariant under a permutation of the particle labels. Therefore, once established for states of the special form (16), relation (15) is also true for $\rho_2 \otimes \rho_{13}$ and for $\rho_1 \otimes \rho_{23}$. Moreover, the left-hand side of (15) is a convex function of ρ. Thus, Eq. (15) holds also for any mixture of the states just mentioned, i.e., for all states in S_3^2.

The quadratic inequality (15) is supplemented by a similar weaker bound for arbitrary states:

$$\forall\ \rho \in S_3: \qquad \langle F_3\rangle_\rho^2 + \langle F_3'\rangle_\rho^2 \le 16. \tag{19}$$

Proof of (19).—

$$\begin{aligned}
\sup(\langle F_3\rangle^2 + \langle F_3'\rangle^2) &= \sup(\langle XC + YC'\rangle^2 + \langle XC' - YC\rangle^2) \\
&= \sup(\langle XC\rangle^2 + \langle YC'\rangle^2 + \langle XC'\rangle^2 + \langle YC\rangle^2 + 2\langle XC\rangle\langle YC'\rangle - 2\langle XC'\rangle\langle YC\rangle) \\
&\le \sup(\langle XC\rangle^2 + \langle YC'\rangle^2 + \langle XC'\rangle^2 + \langle YC\rangle^2) + 2\sup(\langle XC\rangle\langle YC'\rangle - \langle XC'\rangle\langle YC\rangle) \\
&\le 2\sup(\langle X\rangle^2 + \langle Y\rangle^2) + 4\sup|\langle X\rangle\langle Y\rangle| \le 4\sup(\langle X\rangle^2 + \langle Y\rangle^2) = 16,
\end{aligned} \tag{20}$$

where the supremum is over all $\rho \in S_3$, and I have used $\sup\langle XC\rangle \le \sup|\langle X\rangle| \sup|\langle C\rangle| = \sup|\langle X\rangle|$, etc., and $2|\langle X\rangle\langle Y\rangle| \le \langle X\rangle^2 + \langle Y\rangle^2$.

Equality in (19) is again attained for a GHZ state. This shows that (15), in contrast to its two-particle analogy (3), does distinguish between fully entangled states and those that are at most two-partite entangled.

Let us now consider the general case of n spin-$\frac{1}{2}$ particles. Denote the spin observables on particle j, $j = 1, \ldots, n$, as A_j, A_j'. Further, S_n stands for the set of all n-particle states, and S_n^{n-1} for its subset of those states which are at most $n-1$-partite entangled, defined similarly as S_3^2.

The inequalities of Ref. [5] discussed above form part of a recursive chain, constructed as follows:

$$F_n := \tfrac{1}{2}F_{n-1}(A_n + A'_n) + \tfrac{1}{2}F'_{n-1}(A_n - A'_n), \quad (21)$$

where F'_{n-1} is the same expression as F_{n-1} but with all A_j and A'_j interchanged. It is then shown that

$$\forall\ \rho \in S_n^{n-1}: \qquad |\langle F_n \rangle_\rho| \leq 2^{n/2}, \quad (22)$$

$$\forall\ \rho \in S_n: \qquad |\langle F_n \rangle)_\rho| \leq 2^{(n+1)/2}. \quad (23)$$

Recently, Ref. [6] has provided a generalization of the inequalities (4) and (7) to arbitrary n, namely,

$$\forall\ \rho \in S_n^{n-1}: \qquad \langle S_n^\pm \rangle_\rho \leq 2^{n-1}, \quad (24)$$

$$\forall\ \rho \in S_n: \qquad \langle S_n^\pm \rangle_\rho \leq 2^{n-1}\sqrt{2}, \quad (25)$$

where

$$S_{n+1}^\pm := S_n^\pm A_{n+1} \mp S_n^\mp A'_{n+1}. \quad (26)$$

In order to compare these inequalities, note that the recursive relations (21) and (26) imply the following relations between the operators F_n and $S_n^\pm$. When n is odd, and putting $n = 2k + 1$,

$$S_n^\pm = 2^{k-1}[(-1)^{k(k\pm1)/2}F_n \mp (-1)^{k(k\mp1)/2}F'_n]. \quad (27)$$

When n is even, writing $n = 2k$,

$$S_n^\pm = 2^{k-1}(-1)^{k(k\pm1)/2}F^{(\pm)}, \quad (28)$$

where $F^{(+)} := F$ and $F^{(-)} := F'$.

It appears from these relations that the inequalities (22) and (24) are identical when n is even, and independent when n is odd, as we have already seen in the special case of $n = 3$. A similar remark holds for (23) and (25).

Also in the case of n particles, there are quadratic inequalities which strengthen and unify the results just mentioned. First, note that from (27) and (28) we obtain the following identity:

$$\langle S_n^+ \rangle^2 + \langle S_n^- \rangle^2 = 2^{n-2}(\langle F_n \rangle^2 + \langle F'_n \rangle^2). \quad (29)$$

Hence, quadratic inequalities may be expressed by either pair of operators. In the present case, it is convenient to work with the pair $S^\pm$, since the recursive relation (26) is somewhat simpler than (21).

A straightforward generalization of (20) yields

$$\begin{aligned}\sup_{\rho\in S_n} \langle S_n^+ \rangle^2 + \langle S_n^- \rangle^2 &= \sup_{\rho\in S_n} \langle S_{n-1}^+ A_n - S_{n-1}^- \rangle A_n'^2 \\ &\quad + \langle S_{n-1}^- A_n + S_{n-1}^+ A'_n \rangle^2, \\ &\leq 4 \sup_{\rho\in S_{n-1}} \langle S_{n-1}^+ \rangle^2 + \langle S_{n-1}^- \rangle^2, \quad (30)\end{aligned}$$

which, by induction on (19), yields the following bound for arbitrary quantum states:

$$\forall\ \rho \in S_n: \qquad \langle S_n^+ \rangle_\rho^2 + \langle S_n^- \rangle_\rho^2 \leq 2^{2n-1}. \quad (31)$$

Next, consider an n-particle state of the form $\rho = \rho_{1,\ldots,n-1} \otimes \rho_n$. In analogy with (17), we find

$$\begin{aligned}\langle S_n^+ \rangle^2 + \langle S_n^- \rangle^2 &= \langle S_{n-1}^+ A_n - S_{n-1}^- A'_n \rangle^2 + \langle S_{n-1}^- A_n + S_{n-1}^+ A'_n \rangle^2 \\ &= (\langle A_n \rangle \langle S_{n-1}^+ \rangle - \langle A'_n \rangle \langle S_{n-1}^- \rangle)^2 + (\langle A_n \rangle \langle S_{n-1}^- \rangle + (\langle A'_n \rangle \langle S_{n-1}^+ \rangle)^2 \\ &= (\langle A_n \rangle^2 + \langle A'_n \rangle^2)(\langle S_{n-1}^+ \rangle^2 + \langle S_{n-1}^- \rangle^2) \leq 2 \sup_{\rho\in S_{n-1}} \langle S_{n-1}^+ \rangle^2 + \langle S_{n-1}^- \rangle^2 \leq 2^{2n-2}. \quad (32)\end{aligned}$$

As before, this result is extended to all $(n-1)$-partite entangled states by considerations of particle label invariance and convexity. Relation (32) is the n-particle generalization of (14).

Concluding remarks.—The inequalities presented here provide experimentally feasible means of testing whether multiparticle states are fully entangled, in the sense that violation of (32) is a sufficient condition for full entanglement. These conditions may be useful, since, as shown in Ref. [10], some recent experiments that claim to produce such entangled states did not exclude the possibility of lesser entangled states. Note also that, for n even, the test of the quadratic inequality (32) requires the same coincidence measurements for different spin settings as the linear inequalities (22) and (24). Thus, the greater logical strength of the former is not paid for by an increase in experimental difficulty.

Second, a curious aspect of the n-particle inequalities presented here is that they are obtained from a basic quadratic inequality (3) for $n = 2$, which itself, however, does not distinguish between nonentangled and entangled states.

A final remark concerns the relation between testing the entanglement of quantum states and testing quantum mechanics against hidden variable (HV) theories. In analogy to the local HV theories tested by the traditional Bell inequalities, Svetlichny [4] and Svetlichny and Seevinck [6] consider HV theories of n particle systems with partial separability. In such theories, not all particles are assumed to behave locally (separably) with respect to all others, but there is always some subset of particles that behave locally with respect to the particles in another subset (where both subsets are nonempty, of course). These authors show that the inequalities (12) and (24) also characterize the predictions of all partially separable HV theories. By contrast, the quadratic inequalities (15) and (32) reported here do not hold for such theories. The reason for this is that these inequalities rely on the validity of (3) for any two-particle subsystem. However, in a nonlocal HV model for two particles, the Cirel'son inequality, which follows from (3), can be violated. Hence, the inequalities derived here need not hold for such non-quantum-mechanical theories.

For example, it is easy to construct a partially separable HV model for three particles: let the hidden variable λ have only two possible values, and let $AB = AB' = A'B = C = 1$, $A'B' = C' = -1$ for one value of λ, and $AB = AB' = A'B = C = -1$, $A'B' = C' = 1$, for the other. Then one has $\langle S^{\pm}\rangle_{\rm HV} = 4$, in accordance with (4), but violating (15).

I thank George Svetlichny and Michiel Seevinck for fruitful and stimulating discussions.

Appendix: Proof of inequality (3).—The expression $\langle X\rangle^2 + \langle Y\rangle^2 = \langle AB' + A'B\rangle_\rho^2 + \langle AB - A'B'\rangle_\rho^2$ is a convex function of ρ, and so it will be sufficient to consider pure states only. Let $\rho = |\psi\rangle\langle\psi|$. By the Schmidt biorthogonal decomposition theorem we can write

$$|\psi\rangle = p|\phi_1\rangle|\chi_2\rangle - q|\phi_2\rangle|\chi_2\rangle, \qquad (33)$$

where p and q are two positive numbers satisfying $p^2 + q^2 = 1$ and $|\phi_i\rangle$ and $|\chi_j\rangle$ form orthonormal bases in the two-dimensional Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of the two particles, respectively.

Now choose a system of coordinates x_1, y_1, z_1 for the first particle such that $|\phi_1\rangle = |\uparrow\rangle_1$, $|\phi_2\rangle = |\downarrow\rangle_1$, in the z_1 direction, and a similar coordinate system x_2, y_2, z_2 for the other particle such that $|\chi_1\rangle = |\uparrow\rangle_2$, $|\chi_2\rangle = |\downarrow\rangle_2$, in the z_2 direction. Let further $A = \mathbf{a}\cdot\sigma_1$, $B = \mathbf{b}\cdot\sigma_2$, etc., where σ_i denotes the Pauli spin vector in $\mathcal{H}_i$.

In these coordinates, one may write

$$\langle AB\rangle_\psi = -a_{z_1}b_{z_2} - 2pq(a_{x_1}b_{x_2} + a_{y_1}b_{y_2}), \qquad (34)$$

etc., and so, for given $\mathbf{a}, \mathbf{a}', \mathbf{b}, \mathbf{b}'$, the expression $\langle X\rangle^2 + \langle Y\rangle^2$ is a quadratic function of $2pq$. Hence, it will attain its maximum at one end of the range of $2pq$, either $2pq = 0$ or $2pq = 1$. In the former case the state is factorizable and the inequality is trivially satisfied. In the second case we have

$$\langle X\rangle^2 + \langle Y\rangle^2 = (\mathbf{a}\cdot\mathbf{b}' + \mathbf{a}'\cdot\mathbf{b})^2 + (\mathbf{a}\cdot\mathbf{b} - \mathbf{a}'\cdot\mathbf{b}')^2. \qquad (35)$$

Requiring this to be maximal with respect to variations in $\mathbf{a}$, subject to $\mathbf{a}\cdot\mathbf{a} = 1$, shows that $\mathbf{a}$ lies in the plane of $\mathbf{b}$ and $\mathbf{b}'$; similarly $\mathbf{a}'$ lies in this plane.

Now let α, β, γ, and δ denote the angles from $\mathbf{a}$ to $\mathbf{b}$, from $\mathbf{b}$ to $\mathbf{a}'$, from $\mathbf{a}'$ to $\mathbf{b}'$ and from $\mathbf{b}'$ to $\mathbf{a}$, respectively. Then

$$\begin{aligned}\langle X\rangle^2 + \langle Y\rangle^2 &= (\cos\beta + \cos\delta)^2 + (\cos\alpha - \cos\gamma)^2 \\ &= 4\cos^2\left(\frac{\beta+\delta}{2}\right)\cos^2\left(\frac{\beta-\delta}{2}\right) + 4\sin^2\left(\frac{\alpha+\gamma}{2}\right)\sin^2\left(\frac{\alpha-\gamma}{2}\right) \\ &\leq 4\cos^2\left(\frac{\beta+\delta}{2}\right) + 4\sin^2\left(\frac{\alpha+\gamma}{2}\right) = 4, \end{aligned} \qquad (36)$$

since $\alpha + \beta + \gamma + \delta = 2\pi$.

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