Quantum Entropy and Special Relativity

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We consider a single free spin-$\frac{1}{2}$ particle. The reduced density matrix for its spin is not covariant under Lorentz transformations. The spin entropy is not a relativistic scalar and has no invariant meaning.

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The relationship of thermodynamics to relativity theory has been an intriguing problem for many years [1], and it took a new twist when quantum properties of black holes were discovered [2]. In this Letter, we shall investigate a much simpler problem: the relativistic properties of spin entropy for a single, free particle of spin $\frac{1}{2}$ and mass $m > 0$. We show that the usual definition of quantum entropy [3] has no invariant meaning in special relativity.

The reason is that, under a Lorentz boost, the spin undergoes a Wigner rotation [4] whose direction and magnitude depend on the momentum of the particle. Even if the initial state is a direct product of a function of momentum and a function of spin, the transformed state is not a direct product. Spin and momentum appear to be entangled. This is not the familiar type of entanglement which can be used for quantum communication, because both degrees of freedom belong to the same particle, not to distinct subsystems that could be widely separated.

The quantum state of a spin-$\frac{1}{2}$ particle can be written, in the momentum representation, as a two-component spinor,

$$\psi(\mathbf{p}) = \begin{pmatrix} a_1(\mathbf{p}) \\ a_2(\mathbf{p}) \end{pmatrix}, \tag{1}$$

where the amplitudes a_r satisfy $\sum_r \int |a_r(\mathbf{p})|^2 \, d\mathbf{p} = 1$. The normalization of these amplitudes is a matter of convenience, depending on whether we prefer to include a factor $p_0 = \sqrt{m^2 + \mathbf{p}^2}$ in it, or to have such factors in the transformation law as in Eq. (9) below [5]. Following Halpern [6], we shall use the second alternative, because this is the nonrelativistic notation which appears in the definition of entropy. We use natural units: $c = 1$.

Here we emphasize that we consider normalizable states, in the momentum representation, not momentum eigenstates as usual in textbooks on particle physics. The latter are chiefly concerned with the computation of $\langle \mathrm{in}|\mathrm{out}\rangle$ matrix elements needed to obtain cross sections and other asymptotic properties. However, in general, a particle has no definite momentum. For example, if an electron is elastically scattered by some target, the electron state after the scattering is a superposition that involves momenta in all directions.

In that case, it still is formally possible to ask, in any Lorentz frame, what the value is of a spin component in a given direction (this is a legitimate Hermitian operator). We show that the answers to such questions, asked in different Lorentz frames, are not related by any transformation group. The purpose of the present work is to make a first step toward a relativistic extension of quantum information theory. The important issue does not reside in asymptotic properties, but how entanglement (a communication resource) is defined by different observers. Earlier papers on this subject used momentum eigenstates, just as in particle physics [7]. Here we show that radically new properties arise when we consider localized quantum states.

The density matrix corresponding to Eq. (1) is

$$\rho(\mathbf{p}', \mathbf{p}'') = \begin{pmatrix} a_1(\mathbf{p}')a_1(\mathbf{p}'')^* & a_1(\mathbf{p}')a_2(\mathbf{p}'')^* \\ a_2(\mathbf{p}')a_1(\mathbf{p}'')^* & a_2(\mathbf{p}')a_2(\mathbf{p}'')^* \end{pmatrix}. \tag{2}$$

The reduced density matrix for spin, irrespective of momentum, is obtained by setting $\mathbf{p}' = \mathbf{p}'' = \mathbf{p}$ and integrating over $\mathbf{p}$. It can be written as

$$\tau = \frac{1}{2}\begin{pmatrix} 1 + n_z & n_x - in_y \\ n_x + in_y & 1 - n_z \end{pmatrix}, \tag{3}$$

where the Bloch vector $\mathbf{n}$ is given by

$$n_z = \int [|a_1(\mathbf{p})|^2 - |a_2(\mathbf{p})|^2] \, d\mathbf{p}, \tag{4}$$

and

$$n_x - in_y = \int a_1(\mathbf{p}) a_2(\mathbf{p})^* \, d\mathbf{p}. \tag{5}$$

The reduced density matrix τ gives statistical predictions for the results of measurements of spin components by an ideal apparatus which is not affected by the momentum of the particle. The corresponding entropy is [3]

$$S = -\mathrm{tr}(\tau \ln \tau) = -\sum \lambda_j \ln \lambda_j, \tag{6}$$

where

$$\lambda_j = (1 \pm |\mathbf{n}|)/2 \tag{7}$$

are the eigenvalues of τ.

It is well known that ignoring some degrees of freedom usually leaves the others in a mixed state. What is not obvious is that the amount of mixing depends on the Lorentz frame used by the observer. Indeed, consider another observer who moves with a constant velocity with respect to the one who prepared the above state. In the Lorentz frame

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where the second observer is at rest, the same spin-$\frac{1}{2}$ particle has a state

$$\phi(\mathbf{p}) = \begin{pmatrix} b_1(\mathbf{p}) \\ b_2(\mathbf{p}) \end{pmatrix}. \tag{8}$$

The transformation law is [5,6]

$$b_r(\mathbf{p}) = [(\Lambda^{-1}p)_0/p_0]^{1/2} \sum_s D_{rs}[\Lambda, (\Lambda^{-1}p)] a_s(\Lambda^{-1}p), \tag{9}$$

where D_{rs} is the Wigner rotation matrix [4] for a Lorentz transformation Λ (explicitly given in Ref. [6], p. 134).

As an example, consider a particle prepared with spin in the z direction, so that in the Lorentz frame of the preparer $a_2 = 0$. The Bloch vector has only one component, $n_z = 1$, and the spin entropy is zero. When that particle is described in a Lorentz frame moving with velocity β in the x direction, we have, explicitly,

$$b_1(\mathbf{p}) = K[C(q_0 + m) + S(q_x + iq_y)]a_1(\mathbf{q}), \tag{10}$$

$$b_2(\mathbf{p}) = KSq_z a_1(\mathbf{q}), \tag{11}$$

where we have used the following notations: $q_\mu = (\Lambda^{-1}p)_\mu$ is the momentum variable in the original Lorentz frame, $\gamma \equiv (1 - \beta^2)^{-1/2} \equiv \cosh\alpha$,

$$C \equiv \cosh(\alpha/2), \qquad S \equiv \sinh(\alpha/2), \tag{12}$$

and

$$K \equiv [q_0/p_0(q_0 + m)(p_0 + m)]^{1/2}. \tag{13}$$

The new reduced density matrix τ' is obtained as before by integrating over the momenta. Consider, in particular, the case where $a_1(\mathbf{p})$ is a Gaussian (a minimum uncertainty state):

$$a_1(\mathbf{p}) = (2\pi)^{-3/4} w^{3/2} \exp(-\mathbf{p}^2/2w^2). \tag{14}$$

All calculations can be done analytically. To leading order of $w/m \ll 1$, we obtain for the new components of the Bloch vector (defined as above) $n'_x = n'_y = 0$, and

$$n'_z = 1 - \left(w \tanh\frac{\alpha}{2} \Big/ 2m\right)^2. \tag{15}$$

In the new Lorentz frame, the entropy is positive:

$$S \simeq t(1 - \ln t), \tag{16}$$

where $t = w^2 \tanh^2 \frac{\alpha}{2}/8m^2$. (Note that if the momentum has a sharp value, w and t vanish, and therefore the entropy also vanishes, as expected.)

The reduced density matrix τ has no covariant transformation law, except in the limiting case of sharp momenta (only the *complete* density matrix has one). There is an analogous situation in classical statistical mechanics: A Liouville function can be defined in any Lorentz frame [8], but it has no definite transformation law from one frame to another. Only the complete dynamical system has a transformation law [9].

It is important to understand how linearity is lost in this purely quantum mechanical problem. The momenta $\mathbf{p}$ transform linearly, but the law of transformation of spin components depends explicitly on $\mathbf{p}$. When we compute τ by summing over momenta in ρ, all knowledge of these momenta is lost and it is then impossible to obtain τ' by transforming τ. Not only linearity is lost, but the result is not nonlinearity in the usual sense of this term. It is the absence of *any* definite transformation law which depends only on the Lorentz matrix.

Naturally, linearity is still present in a trivial sense. If $\rho = \sum c_j \rho_j$, then likewise $\tau = \sum c_j \tau_j$, and after a Lorentz transformation $\rho' = \sum c_j \rho'_j$, and $\tau' = \sum c_j \tau'_j$. However, even if we know the values of the coefficients c_j, the mere knowledge of the reduced density matrix τ is insufficient to obtain τ' (although the knowledge of the complete density matrix ρ does determine ρ').

In the case investigated above, the entropy computed in the moving frame is larger than the entropy in the original frame, which was zero. This does not mean that a Lorentz transformation always increases the entropy: If we have a particle in the state $b_r(\mathbf{p})$ as the one given above, with a positive entropy, then an observer moving in the $-x$ direction with the appropriate velocity would say that its state is given by $a_s(\mathbf{p})$. For that observer, the entropy is zero. An invariant definition of entropy could be the minimal value of the latter, in any Lorentz frame. (Likewise, the mass of a classical system is defined as the minimal value of its energy, in any Lorentz frame.) Another possibility would be to use the Lorentz frame where $\langle \mathbf{p} \rangle = 0$. It is unlikely that such definitions lead to analytical formulas, although in any particular case the result can easily be obtained by numerical methods.

An interesting problem is the relativistic meaning of quantum entanglement when there are several particles. For two particles, an invariant definition of entanglement would be to compute it in the Lorentz "rest frame" where $\langle \sum \mathbf{p} \rangle = 0$. However, this simple definition is not adequate when there are more than two particles, because there appears a problem of cluster decomposition: Each subset of particles may have a different rest frame. This is a difficult problem, which is beyond the scope of this Letter.

In summary, we have shown that the notion "spin state of a particle" is meaningless if we do not specify its complete state, including the momentum variables. It is possible to formally define spin in any Lorentz frame, but there is no relationship between the observable expectation values in different Lorentz frames.

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