

Statistical Interpretation of the Bekenstein Entropy for Systems with a Stretched Horizon

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For the two-charge extremal holes in string theory we show that the Bekenstein entropy obtained from the area of the stretched horizon has a statistical interpretation as a “coarse graining entropy”: different microstates give geometries that differ near $r = 0$, and the stretched horizon cuts off the metric at $r = b$ where these geometries start to differ.

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Introduction.—Black holes exhibit an intriguing thermodynamics. Gedanken experiments indicate that a black hole has an entropy $S_{\text{Bek}} = \frac{A}{4G}$, where A is the area of the horizon and G is the gravitational constant [1]. The usual principles of statistical mechanics then suggest that there should be e^S microstates of the hole for given macroscopic parameters of the hole. But the metric of the hole appears to be completely determined by its macroscopic parameters (a fact codified in the conjecture “black holes have no hair”) which means that the gravitational theory is unable to exhibit the degeneracy required to account for the entropy.

String theory has in recent years made significant progress in black hole physics. For extremal and near extremal black holes we can count the microstates for a system of branes that carries the same energy and charges as the black hole, and then we find an entropy S_{micro} that equals the Bekenstein entropy $S_{\text{Bek}} = \frac{A}{4G}$ of the corresponding hole [2]. The concept of the anti-de sitter conformal field theory (AdS/CFT) correspondence [3] suggests that S_{micro} counts the states of the hole in a field theory description *dual* to the gravitational description.

This still leaves the following question: If we insist on using the *gravitational* description of the system, then are we supposed to find e^S different states, and if so, where in the geometry would we find the differences that distinguish these states from each other?

We will look at the extremal two charge system in $4 + 1$ noncompact dimensions. The microscopic entropy S_{micro} of the corresponding branes is nonzero. But the horizon area of the (naively constructed) classical metric is zero, so that S_{Bek} appears to be zero. In [4] the two-charge system was studied in $3 + 1$ noncompact directions. Again S_{micro} was nonzero and the classical horizon area was zero. But the curvature diverged near $r = 0$, so it was argued that a “stretched horizon” should be placed at a location $r = r_{\text{stretch}}$ where the curvature becomes string scale. Further, the local temperature of the Hawking radiation becomes of order the Hagedorn temperature at r_{stretch} . Using the area of the stretched horizon to write $S_{\text{Bek}} = A_{\text{stretch}}/4G$, we find that

$$S_{\text{Bek}} \sim S_{\text{micro}}, \quad (1)$$

in accord with the idea that S_{Bek} is always a measure of the entropy of the system.

In $4 + 1$ noncompact dimensions the two-charge extremal system has zero temperature, and so the local temperature is *zero* at all r . But it was shown in [5] that the curvature again becomes string scale at some r_{stretch} , and (1) is satisfied by the corresponding stretched horizon.

While the large curvature at $r < r_{\text{stretch}}$ allows the possibility that the geometry here may be modified by stringy corrections, it does not mean that such corrections *must* occur. By a set of dualities we can map the system to an extremal D1-D5 system, where the near horizon geometry is locally $\text{AdS}_3 \times S^3 \times T^4$. Now the curvature is *constant* at small r ; further, if the geometry were globally of this form, then it would suffer no quantum corrections whatever. In the present case we have a discrete identifications of points in the AdS_3 so it is not immediately obvious what the corrections will be.

Thus the question arises: Is there a way to define the stretched horizon so that we get (1) for all two-charge systems related by duality? Further, does the construction of a stretched horizon have the interpretation of “coarse graining” over different geometries, with the corresponding S_{Bek} reflecting the number of these geometries?

In this Letter we argue that the physics of the stretched horizon should not be thought of in terms of quantum corrections to the naive metric. Instead, we note that while the metrics corresponding to different microstates of the matter system all look the same far from $r = 0$, close to $r = 0$ they are, in fact, all *different*. To coarse grain over such metrics we may therefore truncate the geometry at a radius $r \sim b$ where the geometries start differing from each other. Putting the stretched horizon at $r_{\text{stretch}} = b$ we find that its area gives $S_{\text{Bek}} \sim S_{\text{micro}}$. Thus we find that the Bekenstein entropy of this system (computed from the area of the stretched horizon) has a direct interpretation in terms of a count of different metrics having the same macroscopic parameters. It is also evident that the result holds for all duality related two-charge systems.

The two-charge system.—We consider type IIB string theory compactified on $T^4 \times S^1$; thus we will be considering holes in $4 + 1$ noncompact dimensions. The T^4 has volume $(2\pi)^4 V$ and the S^1 has length $2\pi R$ (we set $\alpha' = 1$).

Following [4] we consider a fundamental string wrapped n_w times around the S^1 , carrying momentum $P = \frac{n_p}{R}$ along the S^1 . These states are supersymmetric, so that their mass is known for all values of the string coupling g . This mass is $M = Rn_w + \frac{n_p}{R}$. The entropy obtained from counting string states with this mass and charges is $S_{\text{micro}} \approx 2\sqrt{2} \pi \sqrt{n_p n_w}$.

We will refer to the above string states as the FP system, where F stands for fundamental string winding and P stands for momentum. By a sequence of S and T dualities we can map this system to one having $n_1 = n_p$ D1 branes wrapped on the S^1 , and $n_5 = n_w$ D5 branes wrapped on $T^4 \times S^1$. We call this latter system the D1-D5 system.

To achieve our desired goal of obtaining entropy by counting geometries we must perform the following steps: (a) We must construct geometries corresponding to different microstates of the FP or D1-D5 system. (b) These geometries may have singularities at small r . In that case we must be satisfied that there is no further degeneracy associated with that singularity, so that the different geometries do indeed count different states of the system. (c) Finally we must locate the radius $r = b$ where the geometries for generic microstates start to differ from each other. We must then compute the area of a stretched horizon placed at this location, and compare the Bekenstein entropy obtained thereby with the count of microstates.

Geometries for different microstates.—To address (a) consider the FP system. The fundamental string wraps the $S^1 n_w$ times before closing, so there are n_w “strands” of the string at any given value of the coordinate y parametrizing the S^1 . The momentum excitation gives traveling waves along this string, creating vibrations in the eight directions transverse to the string. In a generic vibration mode the different strands do not move together—they separate away from each other since the transverse displacement 8-vector \vec{x} satisfies the periodicity $\vec{x}(y + 2\pi R n_w) = \vec{x}(y)$, rather than $\vec{x}(y + 2\pi R) = \vec{x}(y)$.

The metric for a single string carrying momentum is known [6], and the metric for the “multiwound string” having several strands can be computed by superposing harmonic functions describing the individual strands. In the classical limit of large n_w one must then smooth out the strands into a continuous string source. Such metrics for the multiwound string were found in [7] for a special class of vibration profiles, and extended in [8] to generic vibration profiles. For large n_w the vibration profile for a generic mode changes by a very small amount when $y \rightarrow y + 2\pi R$, so the solutions in the classical limit are found to be y independent and a T duality may be performed along the S^1 . Using this and other dualities the solutions are mapped to geometries for the D1-D5 system [8].

The metric for a general D1-D5 bound state is then found to have the form

$$ds_E^2 = \frac{1}{\sqrt{g_1 g_5}} [-(dt - A_i dx^i)^2 + (dy + B_i dx^i)^2] + \sqrt{g_1 g_5} \sum_{i=1}^4 dx_i dx_i, \quad (2)$$

where the functions g_1 , g_5 , and A_i are given in terms of an arbitrary function $\vec{F}(v)$,

$$g_1(\vec{x}) = 1 + \frac{Q_5}{L} \int_0^L \frac{|\dot{\vec{F}}(v)|^2 dv}{|\vec{x} - \vec{F}(v)|^2}, \quad (3)$$

$$g_5(\vec{x}) = 1 + \frac{Q_5}{L} \int_0^L \frac{dv}{|\vec{x} - \vec{F}(v)|^2}, \quad (4)$$

$$A_i(\vec{x}) = -\frac{Q_5}{L} \int_0^L \frac{\dot{F}_i(v) dv}{|\vec{x} - \vec{F}(v)|^2}. \quad (5)$$

We will not give a form of the field B_i here, since it will not be used. In the dual FP system $\vec{F}(v)$ yields the transverse displacement of the vibrating string as a function of the null coordinate $v = t + y$.

If $|\vec{F}(v)| < b$, then for $r \gg b$ the metric has the form [8]

$$ds_E^2 = \frac{1}{\sqrt{g_1 g_5}} [-dt^2 + dy^2] + \sqrt{g_1 g_5} (dr^2 + r^2 d\Omega_3^2), \quad (6)$$

$$g_1(r) = 1 + \frac{Q_1}{r^2}, \quad g_5(r) = 1 + \frac{Q_5}{r^2}.$$

The charges are related by $Q_1 = \frac{Q_5}{L} \int_0^L |\dot{\vec{F}}(v)|^2 dv$.

The location $r \sim b$ will turn out such that a massless particle falling radially down the throat takes a time $\Delta t \sim R\sqrt{n_1 n_5}$ to reach $r \sim b$. For fixed classical charges Q_1, Q_5 this time goes to *infinity* as $\hbar \rightarrow 0$. Thus if we look up to any “classical” distance down the throat then all the geometries look the same [Eq. (6)], so that we see “no hair.”

But around $r \sim b$ the different metrics start to differ from each other. Further, they all have an “end,” which contains a mild singularity along a certain curve—the shape of this curve depends on the chosen microstate. These different geometries are schematically sketched in Fig. 1(b). Each function $\vec{F}(v)$ gives one extremal D1-D5 geometry, which means that for each classical profile of the oscillating string in the FP system there is a classical D1-D5 geometry. We take this to imply that if we quantize the metric of the D1-D5 system we will get one state of the throat for each quantum state of the string in the FP system. This gives us the count of microstates arising from the different possible geometries of the D1-D5 system with given total charges: $S_{\text{micro}} = 2\sqrt{2} \pi \sqrt{n_1 n_5}$.

We now address the requirement (b). In [9] the propagation of a scalar was studied in the geometries corresponding to different states of the D1-D5 system. The CFT dual of these geometries can be described through an “effective

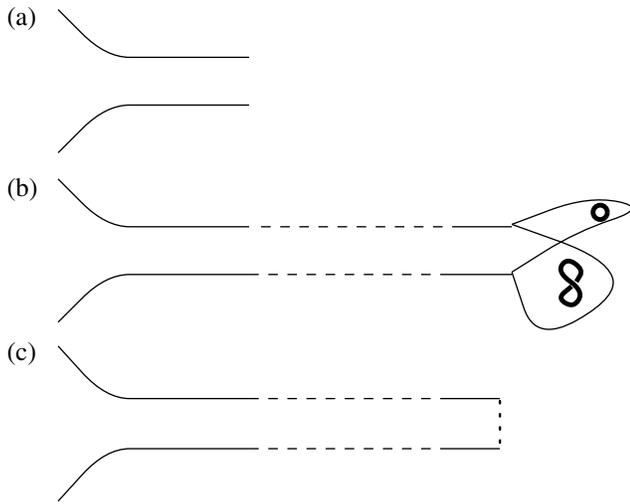


FIG. 1. (a) The geometry for $r \gg b$; (b) two different possible ends to the throat; (c) truncation by a stretched horizon.

string” which carries massless bosonic and fermionic vibration modes as its low energy dynamics. For a class of geometries where the wave packet traveled without distortion it was found that the travel time for the wave packet in the “throat” *exactly* equaled the time in the dual CFT for the vibration modes to travel around the effective string:

$$\Delta t_{\text{CFT}} = \Delta t_{\text{SUGRA}}. \quad (7)$$

The wave packet could not “enter” into the singularity and spend additional time residing at the singularity. It was then argued that for the generic geometry (where the singularity was a curve with more complicated shape) a similar result held: the singularity was mild enough that the wave packet reflected from the singular curve rather than enter into the singular region. Thus we conclude that the throat of the geometry is indeed a dual representation of the D1-D5 CFT with no additional degrees of freedom being associated with the singularity at the end of the throat.

We now perform step (c). We let the geometry be (6) for $r > b$ and put a stretched horizon at the location $r = b$ where the generic geometries start to depart from the form (6) [Fig. 1(c)]. The parameter b will be determined shortly. The area of the stretched horizon (in the 6D geometry) is then

$$A = \int_{r=b} r^3 \sqrt{g_1 g_5} dy d\Omega_3 \approx 4\pi^3 R b \sqrt{Q_1 Q_5}. \quad (8)$$

(We have assumed $b \ll Q_1, Q_5$ which will be well satisfied in the classical limit.) To determine b we start with the *FP* system, and note that the multiwound string has a total length $L_T = 2\pi R n_w$ and carries total momentum $2\pi n_p n_w / L_T$. A statistical analysis shows that the mean wavelength of vibration is $\lambda \sim \tilde{\gamma} L_T$ with $\tilde{\gamma} \sim 1/\sqrt{n_1 n_5}$. If we take a string carrying vibrations with $\lambda \sim \tilde{\gamma} L_T$, then we find upon dualizing to D1-D5 [7]:

$$b \sim \frac{\sqrt{Q_1 Q_5} \tilde{\gamma}}{R} \sim \frac{g}{\sqrt{V} R}. \quad (9)$$

(In this calculation we need to filter out the low energy tail of the energy distribution; the details of this will be discussed elsewhere.) We can now find the entropy of the stretched horizon ($G^{(6)} = \frac{8\pi^6 g^2}{(2\pi)^4 V}$)

$$S_{\text{Bek}} \equiv \frac{A^{(6)}}{G^{(6)}} = \frac{A^{(5)}}{G^{(5)}} \sim \sqrt{n_1 n_5}, \quad (10)$$

which agrees with $S_{\text{micro}} \sim 2\pi\sqrt{2}\sqrt{n_1 n_5}$.

We now extend the above analysis to the case where the two-charge system also carries angular momentum. Let the angular momentum be $J\hbar$ in the x_1 - x_2 plane (these are two of the four noncompact directions). An analysis of the microstate (in the *FP* language) gives $S_{\text{micro}} = 2\pi\sqrt{2}\sqrt{n_1 n_5 - J}$. We then find the geometries for different *FP* microstates with angular momentum J and dualize these to geometries for the D1-D5 system. The metrics are given by (2)–(5) with

$$\vec{F}(v) = a\vec{e}_1 \cos \frac{2\pi v}{L} + a\vec{e}_2 \sin \frac{2\pi v}{L} + \vec{X}(v). \quad (11)$$

The singularity now lies close to a circle of radius

$$a = \frac{g}{\sqrt{V} R} \sqrt{J} \quad (12)$$

in the x_1 - x_2 plane. The generic geometries differ from each other only in a tube around this circle [each state is specified by its own fluctuation profile $\vec{X}(v)$: $|\vec{X}(v)| < b$], so the stretched horizon has the shape of a “doughnut” in the noncompact space x_1, x_2, x_3, x_4 (Fig. 2). Again performing a statistical analysis of the vibrations in the *FP* system, we find $b \sim \frac{g}{\sqrt{V} R}$. All geometries described by the profile (11) look similar outside the doughnut, and the coefficients of the generic metric (2) can be found using (3)–(5):

$$g_1 = 1 + \frac{Q_1}{f_0}, \quad g_5 = 1 + \frac{Q_5}{f_0}, \quad (13)$$

$$A_i dx^i = \sqrt{\frac{J}{n_1 n_5}} \frac{2\sqrt{Q_1 Q_5} a}{f_0(\vec{x} \cdot \vec{x} + a^2 + f_0)} (x_2 dx_1 - x_1 dx_2),$$

$$f_0 = [(\vec{x} \cdot \vec{x})^2 + 2a^2(x_3^2 + x_4^2 - x_1^2 - x_2^2) + a^4]^{1/2}. \quad (14)$$

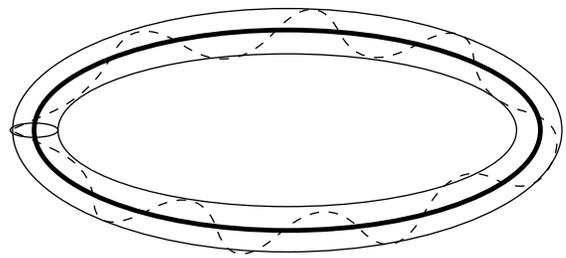


FIG. 2. A typical singular curve (dashed line) and stretched horizon (torus surface) for $J \gg \sqrt{n_1 n_5}$.

The area of the stretched horizon (computed at fixed t) for this metric finally gives

$$S_{\text{Bek}} \equiv \frac{A^{(6)}}{4G^{(6)}} = \frac{VR}{\pi g^2} \int d^3 \Sigma (g_{1g5} - A_i A_i)^{1/2} \\ \sim \sqrt{n_1 n_5 - J} \sim S_{\text{micro}}. \quad (15)$$

Discussion.—Our conventional understanding of entropy is based on coarse graining over a large number of microstates. Black hole entropy has proved puzzling since black holes seem to have no hair and the entropy S_{Bek} is instead given by the horizon area. We have seen above that at least for the two-charge system (which has a stretched horizon rather than a classical horizon) we do indeed have a complete set of hair, that the appearance of the stretched horizon can be regarded as a coarse graining since it truncates the geometries where they start to differ from each other, and that the area of such a stretched horizon gives a S_{Bek} which is of order the entropy S_{micro} found by actually counting the different allowed geometries.

A crucial ingredient in the above result was the fact that the D1-D5 bound states had a nonzero size; this caused the throat to end at some point before reaching $r = 0$, with a metric near the end that reflected the choice of microstate. This nonzero size could itself be traced back, through duality to the FP system, to the fact that a string (F) carrying momentum (P) must spread over a certain transverse region in order to carry the momentum. Thus the nonzero size and the consequent hair are a very basic feature of the structure of two-charge states.

Should we regard the stretched horizon of the two-charge system as a black hole horizon? We performed a detailed investigation (to be presented elsewhere) of the trajectory of a massless particle that falls into the region $r < b$. For a generic microstate the singular curve of the metric (2) is a complicated “random curve” in the transverse coordinates x^1, \dots, x^4 . A null geodesic gets deflected through an angle of order unity on passing near any point on this curve, and as a consequence the particle stays trapped in the region $r < b$ for times that go to infinity in the classical limit (Fig. 3). If we choose to describe the particle by its wave function instead, then we expect large trapping times due to the presence of approximately localized wave functions in the “random potential” arising from the metric at $r < b$. Thus it appears that we should regard the stretched horizon as a horizon, with the time de-

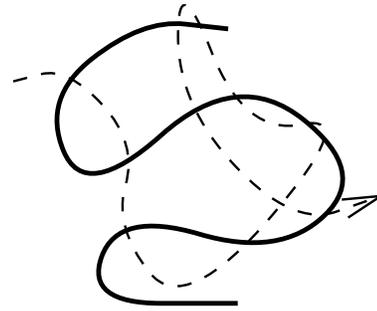


FIG. 3. Typical geodesic (dashed line) near the singular curve (solid line).

lay in emerging from the horizon being due to “trapping” in the hair describing the microstate.

A similar picture may emerge for the D1-D5-momentum hole (which has a classical horizon area). The size of the three-charge bound state at weak coupling was argued to have the same algebraic expression as the horizon radius [10]. The metrics for all microstates are the same up to a “classical distance” down the throat, but the throats may end farther down, with the geometry near the end characterizing the microstate. In view of the fact that the momentum charge generates vibrations within the T^4 we expect that in this case both the S^3 and the T^4 will necessarily be deformed near the end of the throat. Nonextremal holes we regard as highly excited extremal states, and so they are expected to be harder to analyze.

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