

Bell's Theorem for General N -Qubit States

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We derive a single general Bell inequality which is a *sufficient* and *necessary* condition for the correlation function for N particles to be describable in a local and realistic picture, for the case in which measurements on each particle can be chosen between two arbitrary dichotomic observables. We also derive a *necessary* and *sufficient* condition for an arbitrary N -qubit mixed state to violate this inequality. This condition is a generalization and reformulation of the Horodecki family condition for two qubits.

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Local realism imposes constraints on statistical correlations of measurements on multiparticle systems. They are in the form of Bell-type inequalities [1–8]. In a realistic theory the measurement results are determined by “hidden” properties the particles carry prior to and independent of observation. In a local realistic theory the results obtained at one location are independent of any measurements, or actions, performed at spacelike separation. Quantum mechanics predicts violation of these constraints. This is known as Bell's theorem [1].

However, the problems, (a) what are the most general constraints on correlations imposed by local realism and (b) which quantum states violate these constraints, are still open. The latter has been solved in general only in the case of two particles in pure states [9,10] and for two-qubit mixed states [11]. Only recently bounds for local realistic description of a higher-dimensional system have been found in some simple cases [12–14].

Here the answer to the two long-standing questions (a) and (b) is presented for the case of a standard Bell type experiment on N qubits. By a standard Bell experiment we mean one in which each local observer is given a choice between two dichotomic observables. We first derive a *single* general Bell inequality that summarizes all possible local realistic constraints on the correlation functions for a N -particle system. From this inequality one obtains as corollaries the Clauser-Horne-Shimony-Holt (CHSH) inequality [2] for two-particle systems and the Mermin-Ardehali-Belinskii-Klyshko (MABK) inequalities for N particles [4–6]. We show that the correlation functions in a standard Bell experiment can be described by a local realistic model if and only if the general Bell inequality is satisfied. Therefore the general Bell inequality is a *sufficient* and *necessary* condition for correlation functions, in such an experiment, to be describable within a local realistic model. We also find a *necessary* and *sufficient* condition for correlation functions for N qubits in an arbitrary (mixed) quantum state to violate the general Bell inequality in direct measurements. This condition is generalization and reformulation of the one given by the Horodecki family [11] for two qubits.

These results are not only of importance from the fundamental point of view, but also as a research towards identifying ultimate resources for quantum information processing. Recently it was shown [15] that there is a direct link between the security of the quantum communication protocols and the violation of Bell inequalities.

We shall now derive the general Bell inequality. Consider N observers and allow each of them to choose between two dichotomic observables, determined by some local parameters denoted here \vec{n}_1 and \vec{n}_2 . We choose such a notation of the parameters for brevity; of course, each observer can choose independently two arbitrary directions. The assumption of local realism implies the existence of two numbers $A_j(\vec{n}_1)$ and $A_j(\vec{n}_2)$ each taking values $+1$ or -1 , which describe the predetermined result of a measurement by the j th observer of the observable defined by \vec{n}_1 and \vec{n}_2 , respectively (we do not discuss stochastic hidden variable models, as they always can be constructed from underlying deterministic ones). In a specific run of the experiment the correlations between all N observations can be represented by the product $\prod_{j=1}^N A_j(\vec{n}_{k_j})$, with $k_j = 1, 2$. The correlation function, in the case of a local realistic theory, is then the average over many runs of the experiment

$$E(k_1, \dots, k_N) = \left\langle \prod_{j=1}^N A_j(\vec{n}_{k_j}) \right\rangle_{\text{avg}}. \quad (1)$$

The following algebraic identity holds for the predetermined results:

$$\sum_{s_1, \dots, s_N = \pm 1} S(s_1, \dots, s_N) \prod_{j=1}^N [A_j(\vec{n}_1) + s_j A_j(\vec{n}_2)] = \pm 2^N, \quad (2)$$

where $S(s_1, \dots, s_N)$ stands for an arbitrary function of the summation indices $s_1, \dots, s_N \in \{-1, 1\}$, such that its values are only ± 1 , i.e., $S(s_1, \dots, s_N) = \pm 1$. To prove this identity, note that, since $A_j(\vec{n}) = \pm 1$, for each observer j one has either $|A_j(\vec{n}_1) + A_j(\vec{n}_2)| = 0$ and $|A_j(\vec{n}_1) - A_j(\vec{n}_2)| = 2$ or the other way around. Therefore, for all sign sequences of s_1, \dots, s_N the product $\prod_{j=1}^N [A_j(\vec{n}_1) + s_j A_j(\vec{n}_2)]$ vanishes except for just *one* sign sequence, for

which it is $\pm 2^N$. If one adds up all such 2^N products, with an arbitrary sign in front of each of them, the sum is always equal to the value of the only nonvanishing term; i.e., it is $\pm 2^N$.

$$\left| \sum_{s_1, \dots, s_N = -1, 1} S(s_1, \dots, s_N) \sum_{k_1, \dots, k_N = 1, 2} s_1^{k_1-1} \dots s_N^{k_N-1} E(k_1, \dots, k_N) \right| \leq 2^N. \quad (3)$$

Since there are 2^{2^N} different functions $S(s_1, \dots, s_N)$, the inequalities (3) represent a set of 2^{2^N} Bell inequalities for the correlation functions. Many of these inequalities are trivial. For example, when the choice for the function is $S(s_1, \dots, s_N) = 1$ for all arguments, we get the condition $E(1, 1, \dots, 1) \leq 1$. Specific other choices give nontrivial inequalities. For example, for $S(s_1, \dots, s_N) = \sqrt{2} \cos[-\frac{\pi}{4} + (s_1 + \dots + s_N - N)\frac{\pi}{4}]$ one recovers the MABK inequalities, in the form derived by Belinskii and Klyshko [6]. Specifically, for $N = 2$, the well known CHSH inequality [2] follows. For $N = 3$, one obtains the inequality

$$|E(1, 2, 2) + E(2, 1, 2) + E(2, 2, 1) - E(1, 1, 1)| \leq 2. \quad (4)$$

Inequalities, like the one above, with the minus sign at a different location and/or measurements 1 and 2 permuted, form together an equivalence class.

The full set of all 2^{2^N} inequalities (3) is equivalent to the *single* general Bell inequality [16–18]

$$\sum_{s_1, \dots, s_N = -1, 1} \left| \sum_{k_1, \dots, k_N = 1, 2} s_1^{k_1-1} \dots s_N^{k_N-1} E(k_1, \dots, k_N) \right| \leq 2^N. \quad (5)$$

The equivalence of (5) and (3) is evident, once one recalls that for real numbers one has $|a + b| \leq c$ and $|a - b| \leq c$ if and only if $|a| + |b| \leq c$, and writes down a generalization of this property to sequences of an arbitrary length.

Thus far we have shown that when a local realistic model exists, the general Bell inequality (5) follows. The converse is also true: whenever inequality (5) holds one can construct a local realistic model for the correlation function, in the case of a standard Bell experiment. This establishes the general Bell inequality (5) presented above as a *necessary* and *sufficient* condition for a local realistic description of N particle correlation functions in standard

After averaging the expression (2) over the ensemble of the runs of the experiment [compare Eq. (1)], one obtains the following set of Bell inequalities:

Bell-type experiments. This is why one can claim that the set of Bell inequalities (3) is *complete*.

The proof of the sufficiency of condition (5) will be done in a constructive way. A local realistic theory must ascribe certain probabilities to every possible set of predetermined local results. Just as if the local measuring stations were receiving instructions, what should be the measurement results for (here) two possible settings of the local apparatus?

One can ascribe to the set of predetermined local results, which satisfy the following conditions $A_j(\vec{n}_1) = s_j A_j(\vec{n}_2)$, the hidden probability:

$$p(s_1, \dots, s_N) = \frac{1}{2^N} \left| \sum_{k_1, \dots, k_N = 1, 2} s_1^{k_1-1} \dots s_N^{k_N-1} E(k_1, \dots, k_N) \right|, \quad (6)$$

and one can demand that the product $\prod_{j=1}^N A_j(\vec{n}_1)$ has the same sign as that of the expression inside of the modulus defining the $p(s_1, \dots, s_N)$. In this way every *definite* set of local realistic values is ascribed a *unique* global hidden probability. However, if the inequality (5) is not saturated, the probabilities add up to less than 1. In such a case, the “missing” probability is ascribed to an arbitrary model of local realistic noise (e.g., for which all possible products of local results enter with equal weights). The overall contribution of such a noise term to the correlation function is nil. In this way we obtain a local realistic model of a certain correlation function.

However, one should check whether this construction indeed produces the model for the correlation function for the set of settings that enter inequality (5), that is, for $E(k_1, \dots, k_N)$. For simplicity take $N = 2$. One can build a “vector” $(E(1, 1), E(1, 2), E(2, 1), E(2, 2))$ out of the values of the correlation function. The expansion coefficients of this vector in terms of the four orthogonal basis vectors $(1, s_1, s_2, s_1 s_2)$ (recall that $s_1, s_2 \in \{-1, 1\}$) are equal to the expressions within the moduli entering inequality (5). By the construction shown above the local realistic model for $N = 2$ gives the following vector:

$$(E_{LR}(1, 1), E_{LR}(1, 2), E_{LR}(2, 1), E_{LR}(2, 2)) = \frac{1}{4} \sum_{s_1, s_2 = -1, 1} \left[\sum_{k_1, k_2 = 1, 2} s_1^{k_1-1} s_2^{k_2-1} E(k_1, k_2) \right] (1, s_1, s_2, s_1 s_2). \quad (7)$$

Since the vector built out of the correlation function values and its local realistic counterpart have the *same expansion coefficients* in the basis, they are equal. Thus, the sufficiency of (5) as a condition for the existence of a local realistic model is proven. The generalization to an arbitrary N is obvious.

Quantum mechanical predictions can violate the inequality (5). Simply, if a MABK inequality is violated, then the general inequality, which also includes the MABK inequalities, is violated too. However, the converse statement is not always true. The new inequality is more restrictive. In the problem of identifying quantum

states of highly nonclassical traits, it is important to find the class of quantum states, which are not describable by local realistic models. We will now derive the necessary and sufficient condition for an arbitrary (pure or mixed) quantum state to violate the general Bell inequality (5).

An arbitrary mixed state of N qubits can be written down as

$$\rho = \frac{1}{2^N} \sum_{x_1, \dots, x_N=0}^3 T_{x_1 \dots x_N} \sigma_{x_1}^1 \otimes \dots \otimes \sigma_{x_N}^N, \quad (8)$$

where σ_0^j is the identity operator in the Hilbert space of qubit j , and $\sigma_{x_j}^j$ are the Pauli operators for three orthogonal directions $x_j = 1, 2, 3$. The set of real coefficients $T_{x_1 \dots x_N}$, with $x_j = 1, 2, 3$ forms the so-called correlation tensor \hat{T} .

$$\sum_{s_1, \dots, s_N = -1, 1} \left| \left\langle \hat{T}, \sum_{k_1=1}^2 s_1^{k_1-1} \vec{n}_{k_1} \otimes \dots \otimes \sum_{k_N=1}^2 s_N^{k_N-1} \vec{n}_{k_N} \right\rangle \right| \leq 2^N. \quad (11)$$

This inequality can be simplified. For each observer there always exist two mutually orthogonal unit vectors \vec{a}_1^j and \vec{a}_2^j and the angle α_j such that $\sum_{k_j=1}^2 \vec{n}_{k_j} = 2\vec{a}_1^j \cos(\alpha_j + \frac{\pi}{2})$ and $\sum_{k_j=1}^2 (-1)^{k_j} \vec{n}_{k_j} = 2\vec{a}_2^j \cos(\alpha_j + \pi)$. Using the notation $c_{x_j}^j = \cos(\alpha_j + x_j \frac{\pi}{2})$, one can write the inequality (11) as

$$\sum_{x_1, \dots, x_N=1}^2 |c_{x_1}^1 \dots c_{x_N}^N \langle \hat{T}, \vec{a}_{x_1}^1 \otimes \dots \otimes \vec{a}_{x_N}^N \rangle| \leq 1. \quad (12)$$

One can transform this inequality into

$$\sum_{x_1, \dots, x_N=1}^2 c_{x_1}^1 \dots c_{x_N}^N |T_{x_1 \dots x_N}| \leq 1, \quad (13)$$

where $T_{x_1 \dots x_N}$ is now a component of the tensor \hat{T} in a *new* set of local coordinate systems, which among their basis vectors have \vec{a}_1^j and \vec{a}_2^j . The two vectors serve as the unit vectors which define, say, the local directions x and y . The values of $c_{x_j}^j$ enter (13) directly, not as moduli, because without this constraint the maximal value of the left hand side does not change.

We conclude from the above reasoning that the *necessary* and *sufficient* condition for an arbitrary N -qubit state to satisfy the general Bell inequality (5) can be put in the following way. *The correlations between the measurements on N qubits satisfy inequality (5) if and only if in any set of local coordinate system of N observers, and for any set of unit vectors $\vec{c}^j = (c_1^j, c_2^j)$ one has*

$$T_{c_1 \dots c_N}^{\text{mod}} \equiv \sum_{x_1, \dots, x_N=1}^2 c_{x_1}^1 \dots c_{x_N}^N |T_{x_1 \dots x_N}| \leq 1. \quad (14)$$

Let us give a geometric interpretation of (14). Suppose one replaces the components of the correlation tensor $T_{x_1 \dots x_N}$ by their moduli $|T_{x_1 \dots x_N}|$, and builds of such moduli a new tensor \hat{T}^{mod} . Suppose, moreover, one transforms this modified tensor into a new set of local coordinate systems, each of which is obtained from the old one by a rotation within the plane spanned by axes 1 and 2 of the initial coordinates. If this new tensor satisfies constraint (14) for an arbitrary choice of the initial set of local coordinate sys-

The correlation tensor fully defines the N -qubit correlation function:

$$E_{\text{QM}}(k_1, \dots, k_N) = \text{Tr}[\rho(\vec{n}_{k_1} \cdot \vec{\sigma} \otimes \dots \otimes \vec{n}_{k_N} \cdot \vec{\sigma})] \quad (9)$$

$$= \sum_{x_1, \dots, x_N=1}^3 T_{x_1 \dots x_N} (\vec{n}_{k_1})_{x_1} \dots (\vec{n}_{k_N})_{x_N}, \quad (10)$$

where $(\vec{n}_{k_j})_{x_j}$ are the three Cartesian components of the vector \vec{n}_{k_j} . For convenience we shall write down the last expression (10) in a more compact way as $\langle \hat{T}, \vec{n}_{k_1} \otimes \dots \otimes \vec{n}_{k_N} \rangle$, where $\langle \dots, \dots \rangle$ denotes the scalar product in R^{3N} .

We now insert the quantum correlation function $E_{\text{QM}}(k_1, \dots, k_N)$ into the Bell inequality (5), and obtain

tems, then, and only then, a local realistic description of correlation function is possible, in the case of *any* standard Bell experiment.

In other words, $T_{c_1 \dots c_N}^{\text{mod}}$ is a component of T^{mod} along directions defined by the unit vectors \vec{c}^j , $j = 1, \dots, N$. If the condition (14) holds, then the transformed components $\hat{T}_{c_1 \dots c_N}^{\text{mod}}$ do not have values larger than 1. Only then can they describe products of local results, which are only of the values ± 1 , as it is for any correlation tensor. One therefore can express the condition (14) as follows: within local realistic description \hat{T}^{mod} is also a possible correlation tensor. This bears a similarity with the Peres [19] necessary condition for separability (a partially transposed density matrix is a possible density matrix). Note that (14) could also be put in yet another way: arbitrary changes of the signs of some of the coordinates of \hat{T} still leave it as a possible correlation tensor.

By applying the Cauchy inequality to the middle term of expression (14) one obtains directly the following useful and simple *sufficient* condition for the local realistic description of the correlation functions for N qubits. If in *any* set of local coordinate systems of N observers

$$\sum_{x_1, \dots, x_N=1}^2 T_{x_1 \dots x_N}^2 \leq 1, \quad (15)$$

then the correlations between the measurements on N qubits satisfy the general inequality (5).

By performing rotations in the planes defined by directions 1 and 2 of each of the N observers, one can vary the values of the elements of the correlation tensor, but these variations do not change the left-hand side of inequality (15). In this way, one can find local coordinate systems for which some of the correlation tensor elements vanish. Thus the criterion (15) can involve a smaller number of them (compare the three qubit case in Ref. [20]).

There are special situations for which (15) turns out to be both the necessary and sufficient condition for correlation functions to satisfy the general Bell inequality

(5). Formally this arises whenever the two vectors $(c_1^1 \cdots c_1^N, \dots, c_2^1 \cdots c_2^N)$ and $(|T_{11\dots 1}|, \dots, |T_{22\dots 2}|)$ in (13) are parallel. Only then, since the first vector has a unit norm, does the expression on the left hand side of (13) reach the one on the left hand side of (15), and thus the conditions (13) and (15) are equivalent ones.

Let us consider two examples of application of our results. We first study an arbitrary two-qubit state to recover the Horodeckis' condition [11]. In this case, since the two vectors $(|T_{11}|, |T_{12}|, |T_{21}|, |T_{22}|)$ and $(c_1^1 c_1^2, c_1^1 c_2^2, c_2^1 c_1^2, c_2^1 c_2^2)$ in (13) can be made parallel by a suitable choice of free parameters, Eq. (15) is the necessary and sufficient condition for the violation of local realism. Two of the parameters come from the arbitrariness in selecting the two local coordinate systems (i.e., they come from arbitrary transformation of the correlation tensor by rotations within 1–2 planes of each of the two local coordinate systems). Two more parameters are the two angles α_j which define the second vector. Therefore the condition reads $T_{11}^2 + T_{12}^2 + T_{21}^2 + T_{22}^2 \leq 1$. In addition, one can always find local coordinate systems such that $T_{12} = T_{21} = 0$ and our condition transforms into $T_{11}^2 + T_{22}^2 \leq 1$. This is equivalent to the Horodeckis' condition [11], as T_{11}^2 and T_{22}^2 for the diagonalized correlation tensor are equal to two eigenvalues of the matrix $\hat{T}^T \hat{T}$, where \hat{T}^T is the transposed \hat{T} .

As another example, we consider the Werner states. Such states have the form $\rho_W = V|\psi_{\text{GHZ}}\rangle\langle\psi_{\text{GHZ}}| + (1 - V)\rho_{\text{noise}}$, where $|\psi_{\text{GHZ}}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_1 \cdots |0\rangle_N + |1\rangle_1 \cdots |1\rangle_N)$ is the maximally entangled (GHZ) state [3] and $\rho_{\text{noise}} = I/2^N$ is the completely mixed state. Here the weight V of the GHZ state can operationally be interpreted as the interferometric contrast observed in a multiparticle correlation experiment. The nonvanishing components of the correlation tensor in the xy planes for the Werner state are the components which contain an even number of y 's and $T_{xx\dots x}$. There are altogether 2^{N-1} of them. Their values are either $+V$ or $-V$. Since again the two vectors in (13) can be made parallel, Eq. (15) is the necessary and sufficient condition for the violation of local realism. Indeed, if one rotates all but one of the local coordinate systems by 45° , then all 2^N components of the vector $(|T_{11\dots 1}|, \dots, |T_{22\dots 2}|)$ become equal to $V/\sqrt{2}$. Furthermore, if one chooses all α_j equal to $-\pi/4$, the unit vector $(c_1^1 \cdots c_1^N, \dots, c_2^1 \cdots c_2^N)$ has all its components equal to $1/\sqrt{2^N}$. Therefore, the two vectors are parallel. Thus, using criterion (15) we conclude that the correlation functions for the Werner state definitely cannot be described by local realism if and only if $V > 1/\sqrt{2^{N-1}}$.

More applications of the formalism, leading to some unexpected results, are given in [21]. There a family of pure entangled states is found, which *do not violate any*

Bell inequality for correlation functions, for the standard Bell experiment.

It will be interesting to see generalizations of the criteria for violation of local realism to the cases of higher-dimensional systems than qubits and to more measurement choices for each observer than two. One can expect in such cases even stronger restrictions for the local realistic description [22].

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