Fractal Properties of Trivelpiece-Gould Waves in Periodic Plasma-Filled Waveguides

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It is shown that dispersion curves describing a spectrum of Trivelpiece-Gould (TG) waves in periodic plasma-filled waveguides have a fractal nature. They are not solid lines as for other types of waves in periodic waveguides but suffer from discontinuities of the first kind at any $k_z = (P/Q)(2m+1)\pi/d$, where P and Q are integers, d is the period of the corrugation, and m is the transverse index of a mode. The gaps correspond to forbidden bands. The evaluation of the Hausdorf dimension of the dispersion curves is presented. Finally, qualitative consequences of the fractal nature of TG waves for plasma microwave electronics are discussed.

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Investigation of electromagnetic properties of bounded plasma systems is of great scientific and practical interest. In particular, such systems can be exploited in a set of novel technological applications such as elaboration of plasma methods for collective acceleration, high-power microwave generation, transportation of high-current electron beams, etc. Smooth or corrugated cylindrical plasma waveguides are the most widespread bounded plasma systems which can be met in different applications. Electromagnetic properties of smooth cylindrical plasma waveguides at frequencies below the plasma frequency have been studied in detail both theoretically and experimentally [1]. It is well known that the spectrum of the TG modes in the simplest case of a smooth waveguide filled with a strongly magnetized cold collisionless uniform plasma is given by the simple expression $k_z(\omega) = [k^2 + \mu_n^2/|\varepsilon(\omega)|R^2]^{1/2}$, where $\varepsilon(\omega) = 1 - \omega_p^2/\omega^2$, R means radius of the waveguide, μ_n nth root of the zeroth order Bessel function, k_z longitudinal wave number, ω_p plasma frequency, and ω wave frequency. However, the treatment of a corrugated plasma-filled waveguide even in this simplest case meets almost insuperable theoretical and numerical difficulties [2,3]. The generally accepted approach which is traditionally used for periodic media and associated with the field representation as a superposition of a finite number of spatial harmonics seems to be useless even in this simplest case. It yields the so-called dense spectrum [2,3] which contains spurious information [4–7] and leads to the divergence of numerical results. Recently, new approaches have been developed [4–7] which allow us to get rid of spurious solutions and to reduce the problem to the functional equation describing adequately the real spectrum of TG modes in periodic plasma-filled waveguides. Only the simplest case of planar periodic waveguides has been treated and the considerations have been restricted to a quasistatic approximation ($k_z \gg \omega/c$). Extending the new approaches to more realistic situations is possible but before doing that, it seems to be reasonable to perform an exhaustive numerical and analytical analysis of the simplest geometry and to remove some inaccuracies in the analytical and numerical considerations as well as in the physical interpretation of the results obtained in [4,6]. This is the main aim of this Letter.

First following [4,6] consider a planar waveguide of periodically varying width which is loaded with a uniform cold collisionless plasma and embedded in a strong axial static magnetic field (see Fig. 1). The electromagnetic field is assumed to be TM polarized, $(E_x, H_y, E_z) \sim e^{-i\omega t}(\omega < \omega_p)$, electrostatic, and symmetric with respect to the z axis $[E_z(x,z) = E_z(-x,z)]$. Dispersion properties and field distribution of the TG modes in such a structure are described by the functional equation for the axial electric field E_z on the waveguide axis [5–7]:

$$\Psi[z + \varphi(z)][1 + \varphi'(z)] + \Psi[z - \varphi(z)][1 - \varphi'(z)] = 0, \tag{1}$$

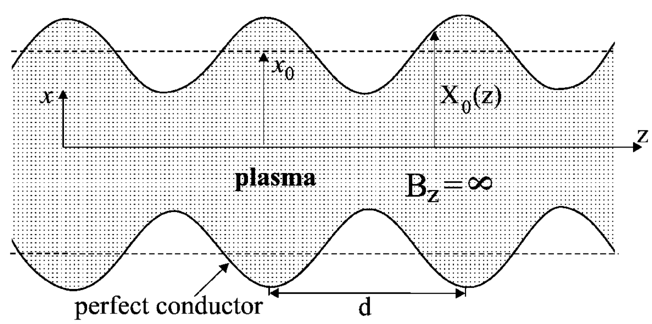

FIG. 1. Geometry of the problem.

where $\Psi(z) = E_z(0,z)$, $\varphi(z) = |\varepsilon(\omega)|^{1/2} X_0(z)$, and $x = X_0(z)$ is the equation of the periodic waveguide boundary: $X_0(z + d) = X_0(z)$, d is the period of the structure. The axial electric field on the waveguide axis is assumed to be quasiperiodic: $\Psi(z + d) = e^{ik_z d}\Psi(z)$. The prime denotes differentiation with respect to z.

It should be mentioned that functional equations of the form of Eq. (1) can be met rather frequently in different fields of theoretical and mathematical physics (see, for example, [8] and the references therein). In contrast to [8], here Eq. (1) should be considered as an eigenvalue/eigenfunction problem. Introducing a new function by the relation $\Psi(z) = \exp[ik_z z + i\theta(z)]$, we arrive at the following equation for the periodic function $\theta(z)$:

$$\theta[z + \varphi(z)] - \theta[z - \varphi(z)] = i \ln \frac{1 + \varphi'(z)}{1 - \varphi'(z)} + (2m + 1)\pi - 2k_z\varphi(z)\,. \tag{2}$$

Note that Eq. (2) characterizes an ordinary spectrum of TG modes in periodic plasma waveguides [without spurious solutions which are inherent in Eq. (1)] and m means number of a mode (see also [6] for details). Furthermore, Eq. (2) can be rewritten in the form

$$\theta[f(\tau)] - \theta(\tau) = i \ln[\dot{f}(\tau)] + (2m + 1)\pi + k_z[\tau - f(\tau)], \tag{3}$$

where $f(\tau)$ is an implicit function:

$$f(\tau) = z + \varphi(z), \qquad \tau = z - \varphi(z)\,, \tag{4}$$

and $\dot{f} = df/d\tau$. The argument ω is here and in the following omitted. Finally, we introduce a periodic function $F(\tau) = f(\tau) - \tau$ and assume that for some ω and τ

$$F(\tau) = nd\,, \tag{5}$$

where n is an integer. When τ varies inside the interval $(0, d)$, the frequency ω which satisfies the condition (5) is continuously varying in the interval $(\omega_{-n}, \omega_{+n})$ due to the continuity of $f(\tau)$. Using (4) we can find that $\omega_{\pm n} = \omega_p/[1 + (nd/X_{0\pm})^2]^{1/2}$, where $X_{0\pm}$ are the maximal and minimal widths of the waveguide. It is easy to show that for $\omega_{-n} < \omega < \omega_{+n}$, Eq. (5) has at least two roots $\tau_{1,2}$ such that $\dot{F}(\tau_1) > 0$ and $\dot{F}(\tau_2) < 0$ due to the periodicity of $F(\tau)$. Taking Eq. (3) at the points $\tau_{1,2}$, where due to the periodicity of $\theta(\tau)$ the left-hand side of Eq. (3) is zero, we come to the relations

$$k_z nd = \pi(2m + 1) - i \ln[\dot{F}(\tau_{1,2}) + 1], \tag{6}$$

which are not satisfied simultaneously at any k_z and ω since $\dot{F}(\tau_1)$ and $\dot{F}(\tau_2)$ are different from each other. Hence the intervals of frequencies $(\omega_{-n}, \omega_{+n})$ can be treated as forbidden bands. At the ends of these intervals, $\tau_1 = \tau_2$ and $\dot{F}(\tau_{1,2}) = 0$. Hence Eq. (5) is satisfied yielding $k_z = k_0(m + 1/2)/n$ with $k_0 = 2\pi/d$.

The obtained forbidden bands have been found numerically in [6,7]. They should be considered as the first-order forbidden bands. It will be shown below that there exists also an infinite number of higher-order forbidden bands which were missed in [6,7] since they were very narrow for those parameters considered in [6,7]. Substituting $f(\tau)$ for τ, we can rewrite Eq. (3) in the form

$$\theta(f(f(\tau))) - \theta(f(\tau)) = i \ln \frac{df(f(\tau))}{df(\tau)} + (2m + 1)\pi + k_z(f(\tau) - f(f(\tau)))\,. \tag{7}$$

Adding Eq. (7) and Eq. (3) yields

$$\theta(f(f(\tau))) - \theta(\tau) = i \ln \frac{df(f(\tau))}{d\tau} + 2(2m + 1)\pi + k_z(\tau - f(f(\tau)))\,. \tag{8}$$

Repeating this procedure s times, we obtain

$$\theta(f^{(s)}(\tau)) - \theta(\tau) = i \ln(\dot{f}^{(s)}(\tau)) + s(2m + 1)\pi + k_z(\tau - f^{(s)}(\tau))\,, \tag{9}$$

where $f^{(s)}(\tau) = \underbrace{f(f(\ldots(f}_{s}(\tau))\ldots)), s = 1, 2, \ldots, \infty$. Introducing a periodic function $F_s(\tau) = f^{(s)}(\tau) - \tau$, we can proceed in the same way as in the analysis for the first-order forbidden bands. Finally, we get equations defining the upper and lower boundary frequencies $\omega^{(s)}_{\pm n}$ for the nth forbidden band of the sth order:

$$F_s(\tau) = nd, \qquad \dot{F}(\tau) = 0\,. \tag{10}$$

The wave numbers corresponding to these frequencies can be derived analytically assuming that the left-hand side of Eq. (9) is equal to zero:

$$k_z = s\,\frac{m + 1/2}{n}\,k_0\,. \tag{11}$$

Hence a dispersion curve for TG modes in a periodic plasma waveguide shows an infinite number of gaps forming an infinite number of stop bands when $k_z d/\pi(2m + 1)$ is a rational number, where m is the transverse index of the TG mode.

Figure 2 shows the curve for the fundamental TG mode ($m = 0$) calculated using Eqs. (10) and (11) for a sinusoidally rippled plasma-filled waveguide: $X_0(z) = x_0(1 + \alpha \cos k_0 z)$. All forbidden bands up to the 40th order were taken into account. The accuracy in the computation of the frequencies was within $10^{-3}\%$. In the inset, the fine structure of the dispersion curve is shown demonstrating its self-similarity. Moreover, dispersion curves for TG modes in periodic plasma-filled waveguides are not ordinary solid lines as for electromagnetic modes. The topological properties of these curves can hardly be characterized by an ordinary topological dimension.

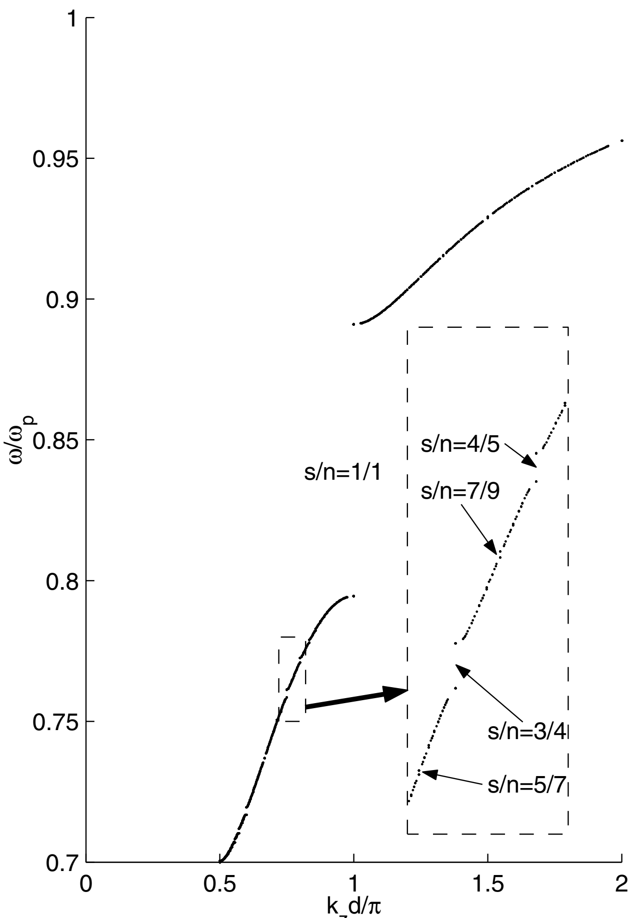

FIG. 2. Dispersion curve for the fundamental TG mode in the case of a sinusoidally rippled plasma-filled waveguide for parameters: $\alpha = 0.2$, $x_0 = 1.4$ cm, $k_0 = 3.67$ cm^{-1}.

Instead they can be characterized by a fractal or Hausdorf dimension D_H which can be calculated by applying the method of covering [9]. According to this method, the whole frequency range $\omega_{\min} < \omega < \omega_p$, where $\omega_{\min}$ corresponds to $\varphi'_{\max} = 1$ [4–7], is covered by equal pieces of length r. Then the passbands of TG modes are plotted. To determine the passbands, we should remove all those forbidden bands which are located inside the range $\omega_{\min} < \omega < \omega_p$. Since the number of passbands as well as the number of forbidden bands is infinite in this interval, the passbands are determined as those frequency intervals which remain after extracting all forbidden bands up to some high but fixed order $s_{\max}$. The number of pieces $N(r)$ which fully or partially overlap with the assumed passbands are then calculated. Then the dependence of $\ln N(r)$ on $\ln(1/r)$ and its derivative are plotted. The results of the calculations for $d[\ln N(r)]/d[\ln(1/r)]$ as a function of $\ln(1/r)$ are shown in Fig. 3(a). As can be seen, this dependence has an interval $[6 < \ln(1/r) < 12]$ where it is almost constant. Increasing the accuracy of the computations leads to a smoother curve in this interval, while increasing $s_{\max}$ leads to an expansion of this interval. The value of $d[\ln N(r)]/d[\ln(1/r)]$ in this interval is equal to the sought quantity D_H. From the results obtained it follows that for the case considered $D_H = 0.968 \pm 0.002$. It is the same for different radial TG modes. When the depth of corrugation decreases, D_H tends to unity and the fractal dispersion curves transform to the ordinary dispersion curves of a smooth plasma-filled waveguide.

The considered method of determination of passbands for TG modes resembles the method of construction of the well-known fractal manifold, the Cantor set [9]. Therefore, for comparison, Fig. 3(b) shows the dependence of $d[\ln N(r)]/d[\ln(1/r)]$ on $\ln(1/r)$ obtained in the same way for the Cantor set at $s_C = 10$ and 11, where s_C is the number of iterative steps for the Cantor set construction and has the same meaning as $s_{\max}$. As we can see, the curves of this dependence are qualitatively very similar in both cases. Moreover, from Fig. 3(b) it follows that $D_H \simeq 0.63$ what is very close to the exact value of D_H which, in this case, can be derived analytically as $D_H = \ln 2/\ln 3$.

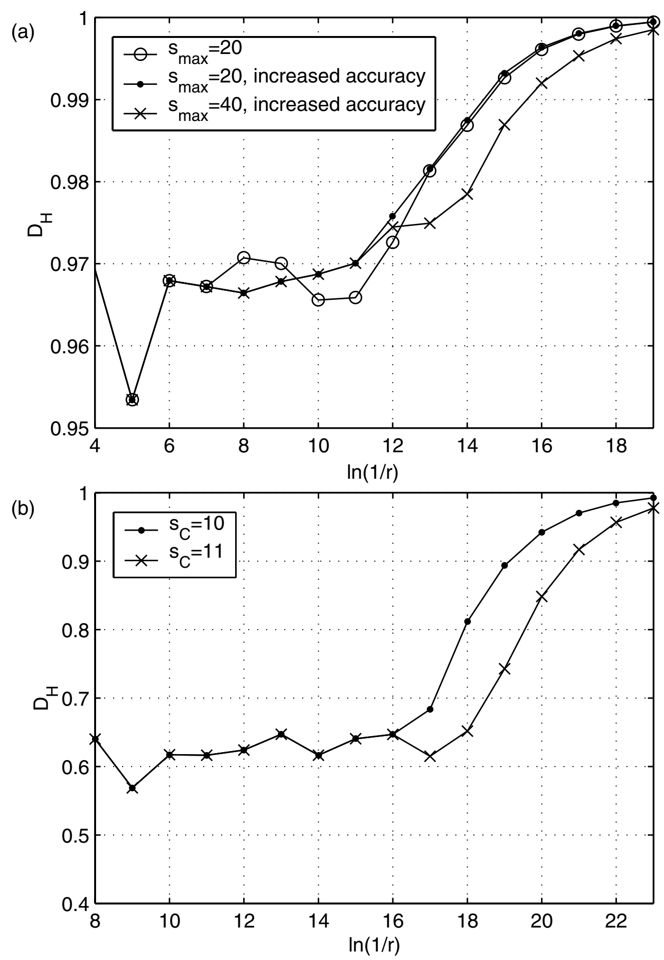

FIG. 3. Dependence of $d[\ln N(r)]/d[\ln(1/r)]$ on $\ln(1/r)$ for (a) a sinusoidally rippled plasma-filled waveguide with the same parameters as in Fig. 2, and (b) for the Cantor set.

The spectral properties of the TG modes were also investigated in [4] on the basis of the theory of circle maps. Some of the properties of the TG modes were correctly predicted and the consideration developed there could be, to some extent, perfectly complementary to that one presented here. In particular, it was shown that, in the bands $\omega_{-n}^{(s)} < \omega < \omega_{+n}^{(s)}$, Eq. (1) can have eigenfunctions in the class of generalized functions corresponding to a constant k_z. Therefore, these bands were interpreted as the bands of spatial mode locking. However, in our opinion, such an interpretation is hardly compatible to the physical nature of the problem under consideration. None of these eigenfunctions with fixed ω and k_z can give a smooth physically meaningful field distribution. This can be formed only by a superposition of them. From the other side, a superposition of them cannot be described by any dispersion relation. Moreover, such oscillations are strongly damped due to spreading and intensive phase mixing. Their asymptotic behavior does not contain undamped or slowly damped terms proportional to $\exp(-i\omega t)$. Since they decay faster than in accordance with the exponential law, they can be referred to as so-called nonproper oscillations according to the terminology accepted for Langmuir waves in a cold nonuniform plasma [10]. Otherwise, this would lead to the existence of waves with infinite group velocity: $v_{gr} = d\omega/dk_z = \infty$ at $\omega_{-n}^{(s)} < \omega < \omega_{+n}^{(s)}$, whereas an interpretation of these bands as forbidden bands is very natural from the physical point of view. Indeed, forbidden bands for waves of arbitrary nature appear in periodic waveguides due to the reflection of the waves from the periodically varying boundary of the waveguide. Moreover, forbidden bands are formed near the intersections of the dispersion curves of the corresponding smooth waveguide with the same dispersion curves but shifted by nk_0 in the ω-k_z plane, where $n = \pm1, \pm2, \ldots \pm\infty$. The following rules hold: the frequency of the intersection point usually lies near the middle of the forbidden band and the wave number of the intersection point coincides with the wave number of the formed gaps. It is easy to verify that the wave numbers of the intersection points exactly coincide with the wave numbers defined by (11), while the frequencies at the intersection points lie between $\omega_{-n}^{(s)}$ and $\omega_{+n}^{(s)}$, i.e., inside the potential stop bands.

The considered fractal properties of the TG modes can have important consequences for understanding the plasma behavior in bounded configurations. The ordinary TG modes which are counterparts of TG modes in a smooth waveguide have a fractal spectrum which is similar to the Cantor set in ω space, i.e., their frequencies lie outside the infinite set of intervals $\omega_{-n}^{(s)} < \omega < \omega_{+n}^{(s)}$. At frequencies $\omega_{\pm n}^{(s)}$, TG modes are not waves in the common sense. Although we assume an infinite length of the periodic plasma-filled waveguide, they behave like proper oscillations in resonators. The frequencies $\omega_{\pm n}^{(s)}$ can be considered as the eigenfrequencies of resonators representing pieces of the plasma-filled periodic waveguide of lengths $2nd/s(2m + 1)$. The field distribution of the TG modes at $\omega_{\pm n}^{(s)}$ is self-organized in such a way that it is periodic on the length $2nd/s(2m + 1)$ and has a structure of a standing wave. The only difference from a common resonator is that the resonance frequencies can lie infinitely close to each other, and that the length of such resonator can vary with changing the resonant frequency. Also it is worth mentioning that the length of the resonators can be shorter than the period of the plasma-filled waveguide d and even tends to zero for large s and m. Such self-organization is possible because a resonant interference occurs at frequencies $\omega_{\pm n}^{(s)}$ between a forward wave and higher spatial harmonics of a backward wave which appear due to reflections from the wall ripples. Therefore, every point of the TG mode spectrum can be considered as a cutoff frequency or as a point lying infinitely close to a cutoff frequency.

Besides, inside the forbidden bands $\omega_{-n}^{(s)} < \omega < \omega_{+n}^{(s)}$, the strongly damped nonproper oscillations with a continuous spectrum can occur [4]. They have no counterparts in smooth plasma-filled waveguides.

The revealed unusual fractal properties of the TG waves can be very essential for the development of an adequate theory of their excitation in periodic plasma-filled structures and treatment of experiments.

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