Two-Component Bose-Einstein Condensates with a Large Number of Vortices

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We consider the condensate wave function of a rapidly rotating two-component Bose gas with an equal number of particles in each component. If the interactions between like and unlike species are very similar (as occurs for two hyperfine states of ^{87}Rb or ^{23}Na) we find that the two components contain identical rectangular vortex lattices, where the unit cell has an aspect ratio of $\sqrt{3}$, and one lattice is displaced to the center of the unit cell of the other. Our results are based on an exact evaluation of the vortex lattice energy in the large angular momentum (or quantum Hall) regime.

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Experiments on rotating Bose gases have progressed rapidly in the last two years. Soon after the pioneer work at JILA [1] and ENS [2], the MIT group created a vortex lattice with as many as 160 vortices [3]. Recently, the JILA group has invented an ingenious method to increase the angular momentum of a condensate by performing evaporative cooling on a rotating normal cloud [4]. In this process, the system spins faster and faster as it is cooled, while remaining close to equilibrium. With such rapid progress, one expects that equilibrium Bose gases with even larger angular momenta may be produced in the near future.

At present, most experiments on vortex lattices are performed in single component Bose systems. It is natural to ask what happens in two-component Bose gases, such as those made up of two hyperfine spin states of the same atom. The vortex lattices in such systems are bound to be more intricate than those in single component condensates, as the vortices in different components can move relative to one another to minimize the energy. The purpose of this paper is to study the vortex lattices of two-component systems with a large number of vortices, in what we call the "mean field quantum Hall regime." This is the regime where mean field theory remains valid so that each component (labeled by an index "i," $i = 1, 2$) is characterized by a condensate wave function Ψ_i; yet the angular momentum of the system is so high that Ψ_i is made up of the orbitals in the lowest Landau level in the plane perpendicular to the rotation axis. It has been shown recently [5] that this regime will emerge in a *three dimensional* Bose gas at sufficiently high angular momenta [6]. We focus on this regime because the wave function in this limit acquires an analytic structure which allows exact evaluation of the energy of a vortex lattice. As a result, it is possible to scan through a wide range of lattice structures which would be impractical for numerical calculations because of the time and the accuracy required. Although not directly applicable to current experiments on vortex lattices (which are performed at lower angular momenta), the physics of the mean field quantum Hall regime is still quite relevant as the vortex lattices in these two regimes are connected continuously to each other.

One special feature of the majority of two-component gases so far studied (notably mixtures of hyperfine states of ^{87}Rb [7] in magnetic traps or ^{23}Na [8] in optical traps) is that the interactions between like species (denoted g_1 and g_2) and unlike species (denoted g_{12}) are within a few percent of each other. Thus, if there are an equal number of bosons in each component, and each feels the same trapping potential, then the two components will be the same size and contain the same density of vortices. In this case, one expects that each component will contain identical vortex lattices, with one lattice displaced relative to the other. While we are mainly interested in the experimentally relevant cases, where $g_1 \sim g_2 \sim g_{12}$, considerable insight is gained by studying vortex lattices as a function of the interactions. Considering the case $g_1 \sim g_2 \neq g_{12}$, we find a wide range of vortex lattice structures as the parameter $\alpha = g_{12}/\sqrt{g_1 g_2}$ is varied. The vortex lattice has a fixed structure over certain intervals of α, while it varies continuously in others. Near the isotropic point $g_1 = g_2 = g_{12}$ each component contains identical rectangular lattices, with one displaced to the center of the unit cell of the other. The aspect ratio of the unit cell changes with α, and is $\sqrt{3}$ when $\alpha = 1$.

The mean field quantum Hall regime.—The condensate wave functions Ψ_1 and Ψ_2 of a two-component rotating Bose gas are determined by minimizing the grand potential $K = E - \Omega L_z - \mu_1 N_1 - \mu_2 N_2$, where E is the energy of the system, Ω is the rotational frequency, L_z is the angular momentum along z, and μ_i $(i = 1, 2)$ are the chemical potentials fixing the number of bosons N_1 and N_2 in each component. For simplicity, we assume identical trapping potentials for each component. We consider a cigar-shaped trap with the symmetry axis z coinciding with the axis of rotation. As discussed in [5], the slow variation of the trapping potential along z allows one to apply a Thomas-Fermi approximation for the z dependence of Ψ_i and write K as $\int dz\, d\mathbf{r}\, \mathcal{K}(\mathbf{r}, z)$,

$$\mathcal{K}(\mathbf{r}, z) = \sum_{i=1,2} \Psi_i^*[h - \mu_i(z)]\Psi_i + \mathcal{V}, \qquad (1)$$

$$h = \frac{1}{2M}\left(\frac{\hbar}{i}\nabla - M\Omega\hat{\mathbf{z}}\times\mathbf{r}\right)^2 + \frac{1}{2}M(\omega_\perp^2 - \Omega^2)r^2, \quad (2)$$

$$\mathcal{V} = \tfrac{1}{2}g_1|\Psi_1|^4 + \tfrac{1}{2}g_2|\Psi_2|^4 + g_{12}|\Psi_1|^2|\Psi_2|^2, \quad (3)$$

with $\mathbf{r} = (x, y)$, $\mu_i(z) = \mu_i - \frac{1}{2}M\omega_z^2 z^2$, $g_i = 4\pi\hbar^2 a_i/M$, $(i = 1, 2)$, and $g_{12} = 4\pi\hbar^2 a_{12}/M$, where a_i and a_{ij} are the s-wave scattering lengths between like and unlike bosons, respectively. As z is treated as a parameter, it is convenient to write $\Psi_i = \sqrt{n_i(z)}\,\Phi_i(\mathbf{r}; z)$, with $\int|\Phi_i(\mathbf{r}; z)|^2 d^2r = 1$. The number constraint $\int d\mathbf{r}\, dz \times |\Psi|^2 = N_i$ becomes $\int n_i(z)\, dz = N_i$.

As Ω approaches $\omega_\perp$, the wave functions Φ_i are made up of the orbitals $u_m(\mathbf{r})$ of the lowest Landau level in the xy plane, $\Phi_i(\mathbf{r}, z) = \sum_{m=0}^{\infty} c_m(z)u_m(\mathbf{r})$, where $u_m(\mathbf{r}) = (2\pi m!)^{-1/2}[(x + iy)/d]^m e^{-r^2/2d^2}$, and $d = \sqrt{\hbar/M\omega_\perp}$. The potential $\mathcal{K}$ then becomes

$$\mathcal{K} = \sum_{i=1,2}\left[\hbar(\omega_\perp - \Omega)\frac{\langle r^2\rangle_i}{d^2} - \mu_i(z) + \hbar\omega_\perp\right] \times n_i(z) + \mathcal{V}, \quad (4)$$

where $\langle r^2\rangle_i = \int r^2|\Phi_i|^2 d^2r$, and

$$\mathcal{V} = \int d^2r\left[\frac{1}{2}\sum_{i=1,2} g_i n_i^2|\Phi_i|^4 + g_{12}n_1n_2|\Phi_1|^2|\Phi_2|^2\right]. \quad (5)$$

As shown in Ref. [5], wave functions in the lowest Landau level (not normalized) can be written as

$$\phi(\mathbf{r}) = \lambda\prod_\alpha (w - a_\alpha)e^{-r^2/2d^2}, \qquad w = x + iy, \quad (6)$$

where λ is an arbitrary constant and $\{a_\alpha\}$ are the zeros of ϕ. If the zeros form an infinite lattice with unit cell size v_c, it is shown in [5] that $|\phi|^2$ is a product of a Gaussian and a function periodic under lattice translation, i.e.,

$$|\phi|^2 = e^{-r^2/\sigma^2}g(\mathbf{r}), \qquad g(\mathbf{r}) = g(\mathbf{r} + \mathbf{R}), \quad (7)$$

where $\mathbf{R} = n_1\mathbf{B}_1 + n_2\mathbf{B}_2$, n_i are integers, and $\mathbf{B}_1$, $\mathbf{B}_2$ are the basis vectors of the lattice. The width σ reflects the number of vortices of the system. It is given by

$$\sigma^{-2} = d^{-2} - \pi v_c^{-1}. \quad (8)$$

The periodicity of $g(\mathbf{r})$ implies $g(\mathbf{r}) = v_c^{-1}\sum_{\mathbf{K}} g_{\mathbf{K}}e^{i\mathbf{K}\cdot\mathbf{r}}$, where $\{\mathbf{K}\}$ are the reciprocal lattice vectors.

In the following, we consider a two-component Bose gas with equal particle numbers and trapping potentials, and with interactions $g_1 \sim g_2 \neq g_{12}$. If $g_1 = g_2$, the two components are identical and we expect each to contain identical vortex lattices, translated with respect to one another. Sufficiently small differences in $g_1 - g_2$ should not change this structure [though changes may occur in the density profiles $n_i(z)$, the parameters of the lattice, and the relative displacement r_0]. This structure persists because, even when $g_1 \neq g_2$, the two components contain equal vorticity, hence an equal density of vortices. The potential energy is minimized by interlacing the two lattices; if the vortex lattice in one component were to deform, the other has to follow to keep the interaction energy at a minimum. We therefore consider *normalized* condensates Φ_1 and Φ_2 with densities

$$|\Phi_1|^2 = (\pi\sigma^2)^{-1}\sum_{\mathbf{K}}\tilde{g}_{\mathbf{K}}e^{i\mathbf{K}\cdot\mathbf{r}}e^{-r^2/\sigma^2}, \quad (9)$$

$$|\Phi_2|^2 = (\pi\sigma^2)^{-1}\sum_{\mathbf{K}}\tilde{g}_{\mathbf{K}}e^{i\mathbf{K}\cdot(\mathbf{r}-\mathbf{r}_0)}e^{-r^2/\sigma^2}, \quad (10)$$

$$\tilde{g}_{\mathbf{K}} = g_{\mathbf{K}}\Bigg/\left(\sum_{\mathbf{K}'} g_{\mathbf{K}'}e^{-\sigma^2\mathbf{K}'^2/4}\right). \quad (11)$$

The wave function is described by variational parameters $n_i(z)$, σ^2, the basis vectors $\mathbf{B}_i$ (which determine the unit cell size v_c), and the relative displacement $\mathbf{r}_0$.

By integrating Eqs. (9) and (10), one sees that up to terms of relative order v_c/σ^2 the cloud's radius is $\langle r^2\rangle_1 = \langle r^2\rangle_2 = \sigma^2$. Defining the quantities I and I_{12} as $\int|\Phi_i|^4 d^2r \equiv I/(\pi\sigma^2)$ and $\int|\Phi_1|^2|\Phi_2|^2 d^2r \equiv I_{12}/(\pi\sigma^2)$, we have

$$I = \sum_{\mathbf{K},\mathbf{K}'}\tilde{g}_{\mathbf{K}}\tilde{g}_{\mathbf{K}'}e^{-\sigma^2|\mathbf{K}+\mathbf{K}'|^2/4}, \quad (12)$$

$$I_{12} = \sum_{\mathbf{K}}\tilde{g}_{\mathbf{K}}\tilde{g}_{\mathbf{K}'}e^{-i\mathbf{K}'\cdot\mathbf{r}_0}e^{-\sigma^2|\mathbf{K}+\mathbf{K}'|^2/4}, \quad (13)$$

and the potential $\mathcal{K}$ takes the form

$$\mathcal{K} = -[\mu(z) - \hbar\omega_\perp - \hbar(\omega_\perp - \Omega)(\sigma^2/d^2)] \times (n_1 + n_2) + (d^2/2\sigma^2) \times (n_1^2g_1I + n_2^2g_2I + 2n_1n_2g_{12}I_{12}). \quad (14)$$

Explicit expressions for the coefficients $g_{\mathbf{K}}$ are derived by introducing the complex representation for the basis vectors, $b_i \equiv (\hat{\mathbf{x}} + i\hat{\mathbf{y}})\cdot\mathbf{B}_i$. The area of the unit cell is then $v_c = i(b_1^*b_2 - b_2^*b_1)/2$. If we orient the lattice so that b_1 is real, i.e., $\mathbf{B}_1 = b_1\hat{\mathbf{x}}$, $\mathbf{B}_2 = b_1(u\hat{\mathbf{x}} + v\hat{\mathbf{y}})$, we then have

$$b_2 \equiv b_1(u + iv), \qquad v_c = b_1^2 v. \quad (15)$$

In the Appendix, we show that *a function Φ in the lowest Landau level describing a regular vortex lattice contained in a cylindrically symmetric cloud will have the form $\phi(\mathbf{r}) = f(w)e^{-r^2/2d^2}$, with $w \equiv x + iy$, and*

$$f(w) = \theta(\zeta, \tau)e^{\pi w^2/2v_c}, \quad (16)$$

where $\zeta = w/b_1 = (x + iy)/b_1$, $\tau = u + iv = b_2/b_1$, and θ is the Jacobi theta function defined as

$$\theta(\zeta, \tau) = \frac{1}{i}\sum_{n=-\infty}^{\infty}(-1)^n e^{i\pi\tau(n+1/2)^2}e^{2\pi i\zeta(n+1/2)}. \quad (17)$$

The density $|\phi|^2$ is therefore of the form Eq. (7), with σ given in Eq. (8), and

$$g(\mathbf{r}) = |\theta(\zeta, \tau)\exp(-\pi y^2/v_c)|^2. \quad (18)$$

The Jacobi theta function has the quasiperiodic properties

$$\theta(\zeta + 1, \tau) = \theta(\zeta, \tau), \tag{19}$$

$$\theta(\zeta + \tau, \tau) = -e^{-i\pi(\tau+2\zeta)}\theta(\zeta, \tau), \tag{20}$$

which implies the periodic property $g(\mathbf{r}) = g(\mathbf{r} + \mathbf{R})$. The Fourier coefficients of $g(\mathbf{r})$ are

$$g_{\mathbf{K}} = (-1)^{m_1+m_2+m_1m_2} e^{-v_c|\mathbf{K}|^2/8\pi}\sqrt{v_c/2}, \tag{21}$$

where $\mathbf{K} = m_1\mathbf{K}_1 + m_2\mathbf{K}_2$, and $\mathbf{K}_i$ are the basis vector of the reciprocal lattice, $\mathbf{K}_1 = (2\pi/v_c)\mathbf{B}_2 \times \hat{\mathbf{z}}$, $\mathbf{K}_2 = (2\pi/v_c)\hat{\mathbf{z}} \times \mathbf{B}_1$, and

$$v_c\mathbf{K}^2 = (2\pi)^2 v^{-1}[(vm_1)^2 + (m_2 - um_1)^2]. \tag{22}$$

Since we work in the limit of large vortex number, the size of the cloud is much larger than the unit cell, i.e., $\pi\sigma^2/v_c \gg 1$. We can therefore ignore all $\mathbf{K} + \mathbf{K}' \neq 0$ terms in Eqs. (12) and (13), since $\sigma^2\mathbf{K}^2 > \pi\sigma^2/v_c$. We then have

$$I = \sum_{\mathbf{K}} \left|\frac{g_{\mathbf{K}}}{g_{\mathbf{0}}}\right|^2, \qquad I_{12} = \sum_{\mathbf{K}} \left|\frac{g_{\mathbf{K}}}{g_{\mathbf{0}}}\right|^2 \cos\mathbf{K}\cdot\mathbf{r}_0, \tag{23}$$

where $g_{\mathbf{K}}$ is given by Eq. (21) and the $\mathbf{K}$ sum is over the integers m_1, m_2. Since the expressions of I and I_{12} in Eq. (23) are independent of σ^2, the minimization of $\mathcal{K}$ in Eq. (14) with respect to σ^2 and n_i becomes very simple. The optimum σ^2, $\mathbf{n} = (n_1, n_2)$, and $\mathcal{K}$ are given by

$$\sigma^2 = d^2[\mu(z) - \hbar\omega_\perp]/[3\hbar(\omega_\perp - \Omega)], \tag{24}$$

$$\mathbf{n}(z) = (2/3)(\sigma^2/d^2)[\mu(z) - \hbar\omega_\perp]\mathbf{G}^{-1}\cdot\mathbf{1}, \tag{25}$$

$$\mathcal{K} = -(1/3)[\mu(z) - \hbar\omega_\perp]\mathbf{1}\cdot\mathbf{n}(z), \tag{26}$$

$$\mathbf{G} = \begin{pmatrix} g_1 I & g_{12}I_{12} \\ g_{12}I_{12} & g_2 I \end{pmatrix}, \qquad \mathbf{1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{27}$$

It is clear from Eqs. (24) through (27) that the solution for the case where $g_1 - g_2 \ll |g_1 + g_2|$ is very close to that of $g_1 = g_2$. The lattice shape (parametrized by $\mathbf{r}_0$, u, and v) enters the grand potential only through the factor $\mathbf{1}\cdot\mathbf{G}^{-1}\cdot\mathbf{1}$. When $g_1 = g_2$ this term is inversely proportional to $J = I + \alpha I_{12}$, and the most favorable lattice is the one that minimizes J.

Summary of results.—It is interesting to compare the two-component case with the single-component case. In the latter system, energy minimization reduces to minimizing I. The only local minimum is the triangular lattice, where $I = 1.1596$; the square lattice is a saddle point with $I = 1.1803$. The minute difference between these values of I makes a simple numerical minimization of (1) challenging and illustrates the utility of the analytic scheme used here.

For a two-component Bose gas, the most favorable lattice minimizes $I + \alpha I_{12}$. In the minimization it is convenient to measure lengths in units of the basis vector $\mathbf{B}_1 = b_1\hat{\mathbf{x}}$, and write complex representation of $\mathbf{B}_2$ and $\mathbf{r}_0$ as $\tau = u + iv = |\tau|e^{i\eta}$ and $r_0 \equiv a + b\tau$, respectively. The phase diagram of the vortex lattice as a function of the ratio $\alpha = g_{12}/g$ is shown in Figs. 1 and 2. The major features

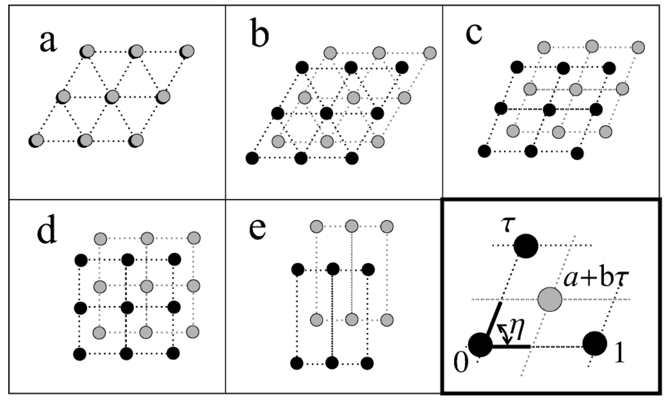

FIG. 1. Phases of the two-component lattice: black and grey dots represent vortices of each of the two fluids. The panels (a) through (e) show the vortex structure in each of the phases described in the text. The final panel depicts the geometry of the lattices; the black and grey dots, respectively, occupy positions in the complex plane $\{m + n\tau\}$ and $\{(a + m) + (b + n)\tau\}$, where m, n are integers. All minimal-energy configurations have $a = b$.

are (a) $\alpha < 0$: In this region the vortices of the two components coincide with each other ($a = b = 0$) to form a triangular lattice ($\tau = e^{i\pi/3}$). (b) $0 < \alpha < 0.172$: The vortex lattice in each component remains triangular. However, r_0 undergoes a first order change so that one lattice is displaced to the center of the triangle of the other ($a = b = 1/3$). The lattice type (characterized by $\tau = e^{i\pi/3}$)

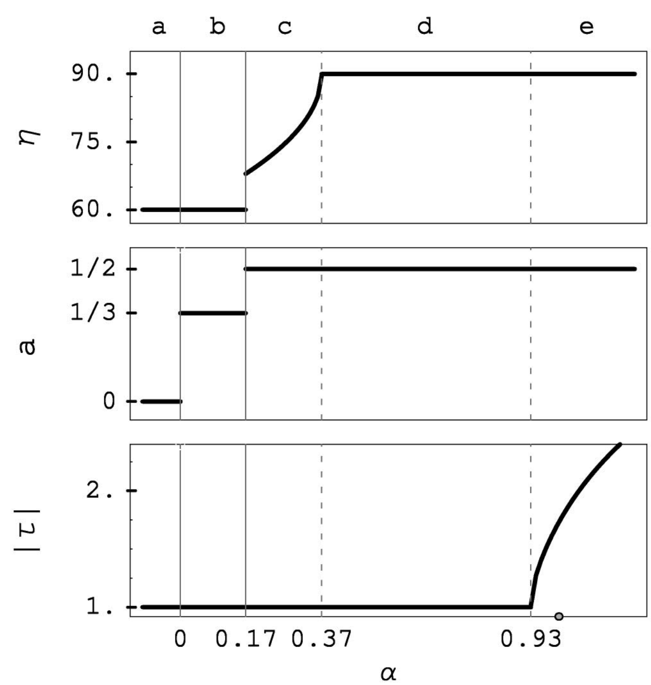

FIG. 2. The parameters of the vortex lattice as a function of $\alpha = g_{12}/\sqrt{g_1 g_2}$, a measure of the importance of interactions between unlike atoms. The phases, labeled (a) through (e) are illustrated in Fig. 1 along with the parameters $\tau = |\tau|e^{i\eta}$ and a. Solid and dashed vertical lines, respectively, denote first and second order phase transitions. The open circle on the horizontal axis indicates $\alpha = 1$.

remains constant within this interval. (c) $0.172 < \alpha < 0.373$: At $\alpha = 0.172$, $\mathbf{r}_0$ jumps from the center of the triangle (i.e., half of the unit cell) to the center of the (rhombic) unit cell ($a = b = 1/2$). The angle η jumps from 60° to 67.95° at $\alpha = 0.172$, and increases continuously to 90° as α increases to 0.372. As a result, the lattice type is no longer fixed and the unit cell is a rhombus. The modulus of τ, however, remains fixed across this region. (d) $0.373 < \alpha < 0.926$: The two lattices are "mode-locked" into a centered square structure throughout the entire interval ($\tau = i, a = b = 1/2$). (e) $0.926 < \alpha$: The lattice type again varies continuously with interaction α. Each component's vortex lattice has a rectangular unit cell ($\eta = \pi/2$) whose aspect ratio $|\tau|$ increases with α. Both ^{87}Rb and ^{23}Na have interaction parameters within this range. At $\alpha = 1$ ($g_1 = g_2 = g_{12} = g$), the aspect ratio is $\sqrt{3}$. If one ignores the difference between the components, the combined lattice is triangular, as is expected.

It is interesting to note that in the absence of rotation, the two components change from miscible to immiscible when α increase beyond 1. No such change, however, happens at $\alpha = 1$ in the high angular momentum limit. This qualitative difference in behavior occurs because the presence of a vortex lattice naturally modulates the density of each component, with the high density regions of one fluid coincident with the low density regions of the other. Thus the system is effectively phase separated whenever staggered vortex lattices are present, even for $\alpha < 1$. In particular, the vortex lattice near $\alpha = 1$ (above or below) is made up of alternating rows of vortices of each component (see Fig. 1), and the system therefore contains stripes in which one component has a high density and the other component has a very low density. As α increases, the stripes become more pronounced.

Final remarks.—The diversity of the vortex lattice structures in the two-component Bose gas has once again demonstrated the rich properties of these systems. Our calculation, based on exact evaluation of the vortex energy, assumes a perfect lattice. Considering the long relaxation times in clouds of dilute atoms, one might see more complicated structures, where patches of vortex domains are separated by defects or grain boundaries. Nevertheless, the underlying equilibrium structure should be reflected within each vortex domain.

So far, we have discussed only two-component systems with simple interpenetrating Bravais lattices. Our method is more general in that it allows the exact evaluation of the energy of an arbitrary regular lattice (with arbitrarily complicated unit cell decoration). Such structures may be favored when the particles in each component have different numbers, trapping potentials, or masses (as in the case of ^{23}Na-^{87}Rb mixtures).

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Appendix.—The general form of a vortex lattice in the lowest Landau level is $\Psi(x, y) = f(w)e^{-r^2/2d^2}$, where $w = x + iy$ and f is an entire function whose zeros form a regular lattice $\{b = nb_1 + n_2b_2\}$, where n_i are integers, and b_1 and b_2 are the complex basis vectors, $[b_2 = b_1(u + iv)]$. Since the Jacobi theta function $\theta[(x + iy)/b_1, u + iv]$ is an entire function with exactly these zeros, we have $f(w) = \theta(\zeta, \tau)h(\zeta)$, where $\zeta = (x + iy)/b_1 = \bar{x} + i\bar{y}$, $\tau = u + iv = b_2/b_1$, and $h(\zeta)$ is an entire function without zeros. To ensure the normalizability of Ψ, this function can be only of the form $h(\zeta) = \exp(c_1\zeta + c_2\zeta^2)$. It is straightforward to show that

$$|\theta(\zeta, \tau)|^2 = \sum_m (-1)^m e^{2\pi i m\bar{x}} e^{-\pi v m^2/2} L_m, \tag{28}$$

$$L_m = \frac{1}{2} \sum_{m'} (1 - e^{i\pi(m+m')}) e^{(i\pi u m - 2\pi\bar{y} - \pi v m'/2)m'} \tag{29}$$

$$= \sqrt{\frac{2}{v}} \sum_k (-1)^{(m+1)k} e^{[-\pi(k+um+2i\bar{y})^2/2v]}, \tag{30}$$

the last line following from the Poisson summation formula. We thus have

$$|\theta(\zeta, \tau)|^2 = \left[\frac{1}{v_c} \sum_{\mathbf{K}} g_{\mathbf{K}} e^{i\mathbf{K}\cdot\mathbf{r}}\right] e^{2\pi y^2/v_c}, \tag{31}$$

where $\mathbf{r} = x\hat{x} + y\hat{y}$, $\mathbf{K} = [2\pi m\hat{x} - 2\pi(n + um)/v\hat{y}]/b_1$, and $g_{\mathbf{K}}$ is given by (21). The density of the system is then $|\Psi(\mathbf{r})|^2 = |\theta(\zeta, \tau)|^2 |e^{c_1\zeta + c_2\zeta^2}|^2 e^{-r^2/d^2}$. For a vortex lattice with inversion symmetry about the origin, $\mathbf{r} = 0$, we have $c_1 = 0$. In addition, if the cloud's envelope is cylindrically symmetric, we have $c_2 = \pi/(2v_c)$, which gives Eqs. (7), (8), (16), and (18). Similar approaches have been used by Tkachenko [9] and Abrikosov [10] in their respective studies of ^{4}He and type two superconductors.

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