Critical Exponents of the Random-Field O(N) Model

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The critical behavior of the random-field Ising model has long been a puzzle. Different methods predict that its critical exponents in D dimensions are the same as in the pure $(D - 2)$-dimensional ferromagnet with the same number of the magnetization components contrary to the experiments and simulations. We calculate the exponents of the random-field O(N) model with the $(4 + \epsilon)$-expansion and obtain values different from the exponents of the pure ferromagnet in $2 + \epsilon$ dimensions. An infinite set of relevant operators missed in previous studies leads to a breakdown of the $(6 - \epsilon)$-expansion.

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The role of quenched disorder in condensed matter depends on its strength, but even weak disorder can strongly modify the nature of the phase transition. This effect is most prominent in the case of random-field (RF) disorder which breaks both the translational symmetry and the symmetry with respect to transformations of the order parameter. In particular, in some systems the arbitrarily weak disorder of this type destroys long-range order [1]. The strong effect of weak disorder makes it difficult to apply the standard perturbative methods of phase transition theory to RF problems. Besides, it is much more difficult to solve exactly a random model than a pure one. As a result, a theory of critical phenomena in the presence of random fields is still absent.

The large number of systems with RF disorder provides a strong motivation to develop such theory. Some of these systems have been known for a long time. Examples are disordered antiferromagnets in the external magnetic field [2], binary liquids in random porous media [3], and vortices in disordered superconductors [4]. Recently a lot of attention was devoted to related problems of disordered liquid crystals [5] and liquid He-3 in aerogels [6]. In contrast to the pure systems even the question of ordering at low temperatures in the presence of random fields is nontrivial. The more difficult problem of the critical behavior is still open in spite of two decades of investigations.

After a lot of controversy the lower critical dimension of the RF Ising model was found [1], but the structure of the phase diagram and the nature of the transition to the ferromagnetic state are still unclear. Different theoretical approaches to the paramagnet-ferromagnet transition predict the dimensional reduction [1]: the critical exponents of the RF O(N) model in D dimensions should be the same as in the pure O(N) model in $D - 2$ dimensions. This prediction relates the three-dimensional RF Ising model to the one-dimensional pure system in contradiction to the presence of long-range order in the former and its absence in the latter. Moreover, the high-temperature expansion [7] shows that the dimensional reduction rule is invalid in any dimension less than the upper critical dimension 6. The most elegant derivation of the dimensional reduction [8] provides an (incorrect) exact solution of the RF O(N) model at zero temperature. The failure of that approach was explained by Parisi on the basis of the complicated energy landscape of the model [9]. However, such an explanation is insufficient for the renormalization group (RG) and $1/N$-expansion. It was conjectured [9] that RG fails because of nonperturbative corrections to the expansion in ϵ. However, such corrections are not found and recent numerical results [7] suggest that the existing predictions for the coefficients of the series in ϵ are wrong [10].

A possible reason why the perturbation theory does not provide satisfactory results is the appearance of some additional relevant operators missed by the existing approaches. This phenomenon is responsible for the failure of the standard RG theory to predict the order of the phase transition in some systems [11]. The possibility of a similar phenomenon in the RF O(N) model was first suggested in Ref. [12], but no new values of the critical exponents were found. Recently a similar idea was used in Ref. [13], but it also did not allow one to calculate the exponents. In the present Letter we demonstrate that there are indeed some additional relevant operators in the problem, although not those found in Ref. [13]. We calculate the critical exponents of the RF O(N) model in $4 + \epsilon$ dimensions and show that they do not satisfy the dimensional reduction. We also demonstrate how the additional relevant operators lead to a breakdown of RG in $6 - \epsilon$ dimensions. Some phenomenological approaches [14] allowed one to obtain the critical exponents different from the dimensional reduction prediction. However, their results based on different unproven assumptions contradict each other. An important breakthrough was made by Mezard and Young [15] who considered the possibility of the replica symmetry breaking in the RF O(N) model at large N. Unfortunately, the approach [15] did not allow the calculation of the exponents. The present Letter contains the first systematic method that explains the failure of RG and allows us to find the critical exponents of the RF systems.

In the critical point the RF O(N) model can be described by the connected and disconnected correlation functions [1]:

$$G_{\rm con}(\mathbf{q}) = \langle [\mathbf{n}(\mathbf{q})\mathbf{n}(-\mathbf{q})] - [\mathbf{n}(\mathbf{q})][\mathbf{n}(-\mathbf{q})]\rangle \sim q^{-2+\eta}; \quad G_{\rm dis}(\mathbf{q}) = \langle [\mathbf{n}(\mathbf{q})\mathbf{n}(-\mathbf{q})]\rangle \sim q^{-4+\bar\eta}, \tag{1}$$

where $\mathbf{n}(\mathbf{q})$ is the Fourier component of the magnetization, the square brackets denote the thermal average, and the angular brackets denote the disorder average. To calculate the critical exponents η and $\bar\eta$ we develop the ϵ-expansion near the lower critical dimension 4 following the line of Ref. [16]. Our starting point is the Hamiltonian of the RF O(N) model

$$H = \int d^D x \left[J \sum_\mu \partial_\mu \mathbf{n}(\mathbf{x}) \partial_\mu \mathbf{n}(\mathbf{x}) - \sum_k [\mathbf{h}_k(\mathbf{x})\mathbf{n}(\mathbf{x})]^k \right], \tag{2}$$

where the unit vector of the magnetization $\mathbf{n}(\mathbf{x})$ has N components and $\mathbf{h}_k$ are random fields with zero average, $\langle h_{k,\alpha}(\mathbf{x}) h_{q,\beta}(\mathbf{y})\rangle = H_k \delta_{\alpha\beta}\delta_{kq}\delta(\mathbf{x}-\mathbf{y})$. The Hamiltonian includes an infinite set of random anisotropies of different ranks. These contributions are allowed by symmetry and turn out to be relevant in the RG sense. The replica Hamiltonian has the form

$$H_R = \int d^D x \left[\sum_a \frac{1}{2T} \sum_\mu \partial_\mu \mathbf{n}_a \partial_\mu \mathbf{n}_a - \sum_{ab} \frac{R(\mathbf{n}_a\mathbf{n}_b)}{T^2} + \ldots \right], \tag{3}$$

where a, b are replica indices, $R(z)$ is some function, T the temperature, and the dots denote the irrelevant terms. Near the zero-temperature fixed point the whole function $R(z)$ is relevant. Indeed, to ensure the fixed length condition $\mathbf{n}_a^2 = 1$ at each RG step, we ascribe the scaling dimension 0 to the magnetization $\mathbf{n}$. The scaling dimension of the temperature is $-2 + O(\epsilon)$. The relevance of any operator is determined by the number of the derivatives in it and the power in which it contains the temperature. This shows that all operators $R_k = \sum_{ab}(\mathbf{n}_a\mathbf{n}_b)^k/T^2$ are relevant in the same space dimensions. The functional RG equations in $4+\epsilon$ dimensions were derived in Ref. [12] (see also [16]). We represent each replica $\mathbf{n}^a(\mathbf{x})$ of the magnetization as a combination of small-scale fields $\phi_i^a(\mathbf{x}), i = 1, \ldots, (N-1)$ and a large-scale field $\mathbf{n}'^a(\mathbf{x})$ of the unit length: $\mathbf{n}^a(\mathbf{x}) = \mathbf{n}'^a(\mathbf{x})\sqrt{1 - \sum_i [\phi_i^a(\mathbf{x})]^2} + \sum_i \phi_i^a(\mathbf{x})\mathbf{e}_i^a(\mathbf{x})$, where the unit vectors $\mathbf{e}_i^a(\mathbf{x})$ are perpendicular to each other and the vector $\mathbf{n}'^a(\mathbf{x})$. The fields ϕ_i change at small scales $a < r < L$, where a is the ultraviolet cutoff, $L \gg a$. The field $\mathbf{n}'$ changes at the scales $r > L$. The RG procedure consists in integrating out the small-scale fields ϕ_i and the rescaling such that the effective Hamiltonian of the field $\mathbf{n}'$ would have the structure (3) with new constants. The RG equation in the first order in ϵ reads

$$\frac{dR(\phi)}{d\ln L} = -\epsilon R(\phi) + [R''(\phi)]^2 - 2R''(\phi)R''(0) - (N-2)\left[4R(\phi)R''(0) + 2\cot\phi R'(\phi)R''(0) - \left(\frac{R'(\phi)}{\sin\phi}\right)^2 \right] + O(R^3, T), \tag{4}$$

where we define $\cos\phi = \mathbf{n}_a\mathbf{n}_b$ to make the equation more compact, and the prime denotes the derivative with respect to ϕ. We are looking for the zero-temperature fixed point $R^*(\phi)$ that describes the phase transition in the RF ferromagnet and satisfies the equation $dR^*(\phi)/d\ln L = 0$. The critical exponents (1) can be expressed [16] via the RG charge $R^*(\phi)$:

$$\eta = -2R''^*(0); \quad \bar\eta = -2(N-1)R''^*(0) - \epsilon. \tag{5}$$

Equation (4) has no fixed points $R^* \sim \epsilon$ analytical in ϕ [12]. This was interpreted [12] as a sign of the strong coupling regime $R^* \sim 1$. We find a nonanalytic weak coupling fixed point $R^* \sim \epsilon$, $R^{IV*}(\phi = 0) = \infty$. If one derives multiloop corrections to Eq. (4) under the assumption of analytic $R(\phi)$ the substitution of our fixed point would lead to an inconsistency since high-order derivatives of R^* enter such a multiloop RG equation. A correct derivation of the higher-order corrections to Eq. (4) is based on the iterative minimization of the Hamiltonian [17] and shows that nonanalytic contributions $\sim\epsilon^{3/2}$ to the critical exponents are possible.

Note that $R''^*(0) < 0$ since $G_{\rm con}(\mathbf{r}) \sim r^{2(N-1)R''^*(0)}$ must be limited. The solution of Eq. (4) can be found numerically with shooting. The region of possible $R''^*(0)$ is limited by the restrictions

$$\epsilon/[2(N-3)] \geq -R''^*(0) = \eta/2 \geq \epsilon/[2(N-2)]. \tag{6}$$

The first inequality follows from the Schwartz-Soffer inequality [18] and is the stability condition for the fixed point [16]. The second one can be derived from the third term of the expansion of the function $R^*(\phi)$ at small ϕ:

$$R^*(\phi) = -\frac{(N-1)R''^*(0)^2}{\epsilon + 4(N-2)R''^*(0)} + \frac{R''^*(0)}{2}\phi^2 \pm \sqrt{\frac{R''^*(0)\epsilon + 2(N-2)R''^*(0)^2}{18(N+2)}}|\phi|^3 + \ldots. \tag{7}$$

Since coefficients of Eq. (4) are singular at $\phi = 0, \pi$, it is difficult to solve Eq. (4) numerically at the vicinity of points 0 and π. I used the expansions of $R^*(\phi)$ in powers of $|\phi|$ and $(\pi - \phi)$ near points 0 and π, respectively. The numerical integration of Eq. (4) was used in the region where the coefficients are not very large. We have solved the RG equation at $N \leq 5$. At $N > 2$ Eq. (4) has exactly one solution compatible with the condition (6).

The critical exponents for those N [19] are

$$\eta(N=3)=5.5\epsilon, \quad \bar{\eta}(N=3)=10.1\epsilon;$$
$$\eta(N=4)=0.79\epsilon, \quad \bar{\eta}(N=4)=1.4\epsilon;$$
$$\eta(N=5)=0.42\epsilon, \quad \bar{\eta}(N=5)=0.70\epsilon.$$

One can see that the critical exponents are different from the dimensional reduction values $\eta(4+\epsilon)=\bar{\eta}(4+\epsilon)=\eta_{\rm pure}(2+\epsilon)=\epsilon/(N-2)$. The incorrect prediction of dimensional reduction in $4+\epsilon$ dimensions [20] could be obtained from Eq. (4) with the wrong assumption that there is an analytical solution $R^*(\phi)$ controlling the critical point.

The second of the inequalities (6) shows that for the *XY* model ($N=2$) there is no appropriate solution of the RG equation. This can be explained by the fact [4] that the RF *XY* model possesses quasi-long-range order in dimension 3. Hence, it has a phase transition in three dimensions. Thus, the critical disorder strength $R^*(\phi)$ is finite in $4+\epsilon$ dimensions and cannot be found with the expansion in powers of infinitesimal ϵ.

We have demonstrated the existence of an infinite set of relevant operators near four dimensions in the RF O(N) model. A possible explanation of the failure of the $(6-\epsilon)$-expansion is thus the appearance of additional relevant operators below some dimension $D<6$. However, we show that another scenario takes place: an infinite set of relevant operators emerges in any dimension less than the upper critical dimension 6. Since the upper critical dimension is the same for the Ising and O(N) models our approach allows us to consider both cases in the same way.

Near six dimensions it is convenient to use the Ginzburg-Landau model with the random field $\mathbf{h}(\mathbf{x})$,

$$H=\int d^Dx\,[(\nabla\mathbf{m})^2+g(\mathbf{m}^2)^2-\mathbf{h}(\mathbf{x})\mathbf{m}], \quad (8)$$

that can be described by the replica Hamiltonian

$$H_R=\int d^Dx\left[\sum_a(\nabla\mathbf{m}_a)^2+\sum_a g(\mathbf{m}_a^2)^2-\Delta\sum_{ab}\mathbf{m}_a\mathbf{m}_b\right], \quad (9)$$

where $\mathbf{m}_a$ are the replicas of the N-component magnetization. The standard power counting suggests that all operators which are relevant in $6-\epsilon$ dimensions are included in Eq. (9). However, if it were so, the dimensional reduction would be correct. The hint as to why the power counting fails is given by the theory of the metal-dielectric transition. It was argued that in the nonlinear sigma model for the metal-dielectric transition one should include an infinite set of relevant operators [21] missed by the power counting. It turns out that the same phenomenon takes place in the RF ferromagnet. As shown below the dangerous operators are $A_k=\sum_{ab}[(\mathbf{m}_a-\mathbf{m}_b)^2]^k$. Since at $k>1$ their canonical dimensions

$$d_c^k=(4-\epsilon)k-6+\epsilon \quad (10)$$

are positive, the power counting predicts that these operators are irrelevant. However, it is important to consider the anomalous dimensions of A_k.

For any eigenoperator A_k of the RG transformation the RG equation has the structure

$$\frac{dA_k}{d\ln L}=[C_0^k+C_1^k\epsilon^s+o(\epsilon^s)]A_k+O(A^2). \quad (11)$$

The power counting is based on the sign of the constant C_0 and predicts that the operator A_k is irrelevant as $C_0^k<0$. However, if the ratio C_1^k/C_0^k grows up to infinity at large k, then for any fixed ϵ there is such k that the correction $C_1^k\epsilon^s$ is greater than C_0^k. Hence, the role of the operators A_k depends on the signs of C_1^k. In the ϕ^4 theory without the random field this sign is negative for any operator which is expected to be irrelevant from the power counting [22]. This agrees with the success of the RG approach in that problem. On the other hand, we shall see that the signs of these constants are positive for some operators in the RF problem. This signals that additional relevant operators emerge.

For the pure ϕ^4 model one can define the rank of an operator with q derivatives and the pth power of the order parameter as $r=p+q-4$. Since only the operators of the same rank can mix [23], for the calculation of the anomalous dimension $C_1^k\epsilon^s$ of an operator with rank r_k one can ignore the diagrams which produce the operators of the higher ranks. It is easy to check that the definition of the rank has to be modified in the RF problem to ensure that the operators of different ranks do not mix: $r=p+q+2t-6$, where t is the number of the different replica indices in the operator.

To avoid the difficult problem of operator mixing we guess a set of relevant operators which do not mix with the other operators of the same rank up to the second order in ϵ. It turns out that the formerly introduced operators $A_k=\sum_{ab}[(\mathbf{m}_a-\mathbf{m}_b)^2]^k$ are relevant and do not mix with the other. At $k>1$ their canonical dimensions (10) are positive and proportional to k. We shall see that the anomalous dimensions are negative and proportional to $(k\epsilon)^2$. If one imposes the fixed length restriction $\mathbf{m}^2=1$ so that the Ginzburg-Landau model reduces to the O(N) model, then the operators A_k can be represented as the linear combinations of the random anisotropies $B_k=\sum_{ab}(\mathbf{m}_a\mathbf{m}_b)^k$. This is natural since the operators B_k are relevant in $4+\epsilon$ dimensions as one could see above.

To the first order in the quartic vertex g the anomalous dimensions of the operators A_k are zero. To prove this

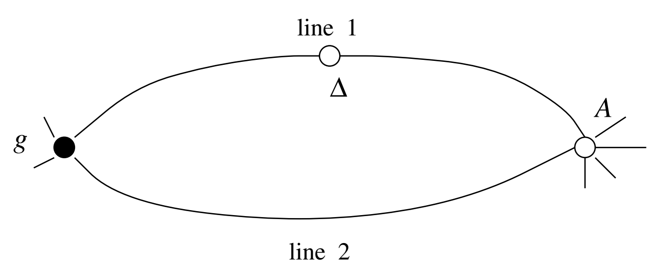

FIG. 1. A first order diagram contributing to the anomalous dimension.

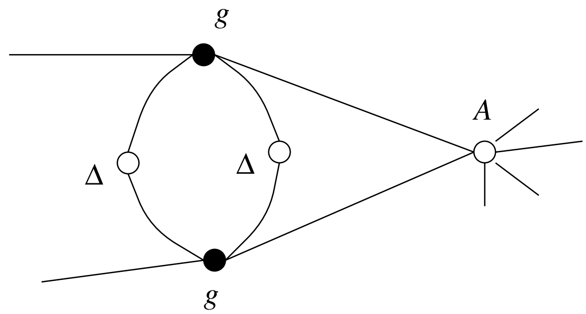

FIG. 2. A second order diagram contributing to the anomalous dimension.

we demonstrate that all diagrams with one vertex gm_a^4 and one vertex $A_k^{ab} = [(\mathbf{m}_a - \mathbf{m}_b)^2]^k$ either are equal to zero or produce operators of higher ranks. Indeed, any line of the Feynman diagram corresponds to the product of a momentum-dependent factor and two operators which differentiate the vertices at the ends of the line with respect to replicas $\mathbf{m}_a$ of the magnetization. For example, line 2 in Fig. 1 acts on the vertex A_k^{ab} as the differential operator $\partial_2 = \partial/\partial\mathbf{m}_a$. Line 1 in which the vertex Δ is inserted acts on the vertex A_k^{ab} as the operator $\partial_1 = \partial/\partial\mathbf{m}_a + \partial/\partial\mathbf{m}_b$. Obviously, $\partial_1 A_k^{ab} = 0$. Hence, any diagram including line 1 is equal to zero. On the other hand, any diagram with one vertex g, one vertex A_k, and without lines in which Δ is inserted produces an operator of a higher rank. Figure 2 shows the only nonzero diagram of the order g^2 that does not increase the rank of the operators A_k. Calculating this diagram one obtains the anomalous dimension

$$d_{\mathrm{an}}^{k,N} = k(N + 2)\epsilon^2/2(N + 8)^2 - \epsilon^2[Nk(2k + 1) + 16k^2 - 10k]/2(N + 8)^2. \quad (12)$$

The anomalous dimension (12) is negative and proportional to k^2. Thus, at any fixed ϵ one expects that the operators A_k with $1/\epsilon^2 \lesssim k$ are relevant but missed by the existing theoretical methods. Although the rigorous analysis requires consideration of all orders in ϵ, the appearance of an infinite set of relevant operators in $6 - \epsilon$ dimensions is plausible since such a set exists near four dimensions and since the alternative explanation [9] of the failure of RG due to nonperturbative corrections is hardly compatible with the existing numerical results [7].

In conclusion, we have calculated critical exponents of the O(N) model in $4 + \epsilon$ dimensions and demonstrated that they do not obey the dimensional reduction. The failure of the dimensional reduction is related to the appearance of an infinite set of relevant operators.

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