Dual Neutral Variables and Knot Solitons in Triplet Superconductors

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We derive a dual presentation of a free energy functional for spin-triplet superconductors in terms of gauge-invariant variables. The resulting equivalent model in ferromagnetic phase has a form of a version of the Faddeev model. This allows one, in particular, to conclude that spin-triplet superconductors allow formation of *stable* finite-length closed vortices (knotted solitons).

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In this Letter we discuss spin-triplet superconductors and show that under certain conditions these superconductors should allow formation of stable finite-length solitons. The discussions of the stable finite-length solitons in 3 + 1 dimensions have a long history [1–8]. This concept was first discussed in mathematical physics: In [1] Faddeev introduced a version of nonlinear O(3) σ-model which allows formation of the stable finite-length solitons which may have a form of a knot or a vortex loop. The stability of these defects in the Faddeev model is ensured by a higher-order derivative term:

$$H = \int d\mathbf{r}\left[\alpha|\partial_k \vec{\mathbf{n}}|^2 + \frac{1}{4e^2}(\vec{\mathbf{n}} \cdot \partial_i \vec{\mathbf{n}} \times \partial_j \vec{\mathbf{n}})^2 \right], \quad (1)$$

where $\vec{\mathbf{n}}$ is a three-component unit vector.

Recently it was realized that versions of this model are potentially relevant for description of many different physical systems ranging from infrared limit of QCD to superconductivity in transition metals [1–8]. Relevance of this model for condensed matter physics was pointed out in Ref. [2] where it was found that in two-band superconductors there exists a mapping between a two-flavor Ginzburg-Landau (GL) functional and a version of a O(3)-symmetric Faddeev model. The knotted solitons are much more complex and structurally complicated topological defects than Abrikosov vortices, and thus its realization in superconductors should open an exceptionally wide range of possibilities of studies of various phenomena associated with them. The remarkable circumstance is that the studies of the properties of these defects in superconductors may result in a "feedback" for the discussions of properties of possible similar defects in the infrared limit of QCD where it was argued that it describes glueballs [3].

So far the two-band superconductors were the only known condensed matter system which is described by a version of the Faddeev model and allows formation of the knotted solitons [2]. In this paper we discuss presentation of the GL functional for a *single-condensate spin-triplet superconductor* in gauge-invariant variables by means of an *exact* duality mapping which also leads to another version of the Faddeev model in the ferromagnetic state of the condensate. A remarkable fact which we observe below is that the resulting effective model for the ferromagnetic spin-triplet superconductor is surprisingly similar to the model discussed in [2], albeit the physical origin of this model in spin triplet superconductors is *principally* different from the model in [2].

We emphasize that in a *charged* spin-1 Bose condensate the stable knotted solitons (in which energy is a nonmonotonic function of its size) are the counterpart of the Volovik-Mineev vortices characterized by a nontrivial Hopf invariant in a *neutral* spin-1 superfluid (discussed in the pioneering papers on superfluid ^{3}He [9–12]). The defects considered in [9–12] have similar topology, but its energy is proportional to its size. Thus, these defects are unstable against shrinkage.

We begin with reminding the reader of some important features of the *neutral* spin-1 condensate following the paper [13]. We write the order parameter of the spin-1 Bose condensate as $\Psi_a(\mathbf{r}) = \sqrt{n}(\mathbf{x})\zeta_a(\mathbf{x})$ where $(a = 1, 0, -1)$ with ζ being a normalized spinor $\zeta^\dagger \cdot \zeta = 1$. Then the energy functional for a neutral spin-1 system is [13]

$$K = \int d\mathbf{r}\, \frac{\hbar^2}{2M}(\nabla\sqrt{n})^2 + \frac{\hbar^2}{2M} n(\nabla\zeta)^2 - \mu n + \frac{n^2}{2}[c_0 + c_2\langle \mathbf{F}\rangle^2], \quad (2)$$

where $\langle \mathbf{F}\rangle = \zeta_a^* \mathbf{F}_{ab}\zeta_b$ is spin. Degenerate spinors are related to each other by gauge transformation $e^{i\theta}$ and spin rotations $\mathcal{U}(\alpha, \beta, \tau) = e^{-iF_z\alpha}e^{-iF_y\beta}e^{-iF_z\tau}$, where (α, β, τ) are the Euler angles. The ground state structure of $\Psi_a(\mathbf{r})$ can be found by minimizing the energy with fixed particle number [13]. Below we shall be mainly interested in ferromagnetic state. This state emerges when $c_2 < 0$. The energy is minimized by $\langle \mathbf{F}\rangle^2 = 1$ and the ground state spinor and density are [13]

$$\zeta = e^{i\theta}\mathcal{U}\begin{pmatrix}1\\0\\0\end{pmatrix} = e^{i(\theta-\tau)}\begin{pmatrix} e^{-i\alpha}\cos^2\frac{\beta}{2} \\ \sqrt{2}\cos\frac{\beta}{2}\sin\frac{\beta}{2} \\ e^{i\alpha}\sin^2\frac{\beta}{2}\end{pmatrix},$$

$$n^o(\mathbf{r}) = \frac{1}{c_0 + c_2}\mu. \quad (3)$$

From this equation it is seen that in the ferromagnetic case there exists an equivalence between gauge transformation

θ and spin rotations τ so the symmetry group is SO(3) [13]. Given the expression for the ground state spinor (3) one can immediately derive the superfluid velocity for a neutral spin-1 ferromagnetic Bose system [13]:

$$\mathbf{v} = \frac{\hbar}{M}[\nabla(\theta - \tau) - \cos\beta\nabla\alpha]. \tag{4}$$

Let us now turn to a *charged* spin-1 Bose condensate. We have the following expression for the free energy of the spin-1 superconductor in the "ferromagnetic" state:

$$F = \int d\mathbf{r}\left[\frac{\hbar^2}{2M}(\nabla\sqrt{n})^2 + \frac{\hbar^2 n}{2M}\left|\left(\nabla + i\frac{2e}{\hbar c}\mathbf{A}\right)\zeta_a\right|^2 - \mu n + \frac{n^2}{2}[c_0 + c_2] + \frac{\mathbf{B}^2}{8\pi}\right] \tag{5}$$

with ground state spinor and density being given by (3). Consequently, the equation for the supercurrent is

$$\mathbf{J} = \frac{i\hbar en}{M}(\zeta_a^*\nabla\zeta_a - \zeta_a\nabla\zeta_a^*) - \frac{4e^2 n}{Mc}\mathbf{A} = \frac{2\hbar en}{M}[\nabla(\theta - \tau) - \cos\beta\nabla\alpha] - \frac{4e^2 n}{Mc}\mathbf{A}. \tag{6}$$

From this equation it is seen that the supercurrent depends not only on phase gradients but also on spin texture.

However, the properties of the charged spin-1 condensate (in particular, topological defects allowed by the system) are principally different from the neutral case that can be seen if we eliminate gauge field by a duality transformation to neutral variables which explicitly shows physical degrees of freedom in the system. Such a mapping, which is actually very different compared to the problem discussed in [2], is discussed below.

We emphasize that in the charged case (5) the free energy features a contribution from magnetic field $\mathbf{B}^2/8\pi$ which can be external or self-induced or both. We also stress that we consider below a superconductor in a simply connected space; that is, our defects do not feature zeros of the order parameter. In this case taking curl from both sides of (6) [and taking into account that in a simply connected space for a regular function holds identity $\mathrm{curl}\nabla(\theta - \tau) = 0$], we arrive at the following equation for the magnetic field in triplet superconductor:

$$B_k = -\frac{c}{4e}[\nabla_i C_j - \nabla_j C_i] + \frac{\hbar c}{4e}(\vec{\mathbf{s}}\cdot\nabla_i\vec{\mathbf{s}}\times\nabla_j\vec{\mathbf{s}}), \tag{7}$$

where $\nabla_i = \frac{d}{dx_i}$ and we introduced the following notations:

$$\vec{\mathbf{s}} = (\sin\beta\cos\alpha, \sin\beta\sin\alpha, \cos\beta); \tag{8}$$

$$\vec{C} = \frac{M}{en}\mathbf{J}. \tag{9}$$

Let us rearrange terms in the GL functional (5). First let us rewrite the second term in (5) as follows:

$$\frac{\hbar^2 n}{2M}\left|\left(\nabla + i\frac{2e}{\hbar c}\mathbf{A}\right)\zeta_a\right|^2 = \frac{\hbar^2 n}{2M}\left[\nabla\zeta_a\nabla\zeta_a^* + \frac{2e}{\hbar c}\mathbf{A}\mathbf{j} + \left(\frac{2e}{\hbar c}\mathbf{A}\right)^2\right] = \frac{\hbar^2 n}{2M}\left[\nabla\zeta_a\nabla\zeta_a^* + \left(\frac{\mathbf{j}}{2} + \frac{2e}{\hbar c}\mathbf{A}\right)^2 - \frac{\mathbf{j}^2}{4}\right] = \frac{\hbar^2 n}{2M}\left[\nabla\zeta_a\nabla\zeta_a^* - \frac{\mathbf{j}^2}{4}\right] + \frac{n}{8M}\vec{C}^2, \tag{10}$$

where

$$\mathbf{j} = (i\zeta_a^*\nabla\zeta_a - i\zeta_a\nabla\zeta_a^*). \tag{11}$$

Then we observe the following circumstances:

$$\nabla\zeta_a\nabla\zeta_a^* = \frac{1}{2}(\nabla\beta)^2 + [\nabla(\theta - \tau - \alpha)]^2\left[\frac{1}{2} + \frac{\cos\beta}{2}\right]^2 + [\nabla(\theta - \tau + \alpha)]^2\left[\frac{1}{2} - \frac{\cos\beta}{2}\right]^2 + [\nabla(\theta - \tau)]^2\frac{\sin^2\beta}{2}.$$

From the above expression it follows that

$$\nabla\zeta_a\nabla\zeta_a^* - \frac{\mathbf{j}^2}{4} = \frac{1}{2}(\nabla\beta)^2 + \frac{\sin^2\beta}{2}(\nabla\alpha)^2 = \frac{1}{2}(\nabla\vec{\mathbf{s}})^2. \tag{12}$$

Now we can express the Ginzburg-Landau functional for spin-1 charged ferromagnetic Bose condensate in the form involving only gauge-invariant variables (compare with [2]). That is, we eliminate the gauge field $\mathbf{A}$ and the variable $\theta - \tau$ in favor of the gauge-invariant variables $\vec{\mathbf{s}}$ and $\vec{C}$. With it Eq. (5) becomes

$$F = \int d\mathbf{r}\left[\frac{\hbar^2}{2M}(\nabla\sqrt{n})^2 + \frac{\hbar^2 n}{4M}(\nabla\vec{\mathbf{s}})^2 + \frac{n}{8M}\vec{C}^2 + \frac{\hbar^2 c^2}{128\pi e^2}\left(\frac{1}{\hbar}[\nabla_i C_j - \nabla_j C_i] - \vec{\mathbf{s}}\cdot\nabla_i\vec{\mathbf{s}}\times\nabla_j\vec{\mathbf{s}}\right)^2 - \mu n + \frac{n^2}{2}[c_0 + c_2]\right]. \tag{13}$$

The above expression is a version of the Faddeev nonlinear O(3) σ-model (1) introduced in [1]. The remarkable circumstance is that this effective model for a spin-triplet superconductor is very similar to the effective model for two-band superconductors where Cooper pairs have spin-0, but the system possesses a hidden O(3) symmetry [2]. The resulting model explicitly displays only physically relevant degrees of freedom and indeed does not depend on θ, τ, and $\mathbf{A}$. The new variables are $n \equiv |\Psi|^2$, the vector $\vec{\mathbf{s}}$ [where the position on the unit sphere S^2 is characterized by the angles α and β (9)] and the massive vector field $\vec{C}$.

The effective action contains a fourth-order derivative term for the vector field $\vec{\mathbf{s}}$. The field $\vec{\mathbf{s}}$ is also coupled to the massive vector field $\vec{C}$. Thus, we can see that the properties of the charged spin-1 condensate are very different from the neutral condensate (2). The effective model (13) displays features of spin-triplet superconductors which can be easily overlooked in the presentation of the effective functional in the form (5). In particular, from (13) we can conclude that the system allows *stable* knotted solitons characterized by a nontrivial Hopf invariant where the stability is ensured by the fourth-order derivative term (the Faddeev term). This term is not explicitly present in the free energy functional in the Ginzburg-Landau form (5). It stems from the term $\mathbf{B}^2/8\pi$ and has the physical meaning of the *self-induced* magnetic field in the presence of a nontrivial spin texture (in this paper we discuss a system with no applied external fields). Indeed, this effect is possible due to the feature of the ferromagnetic state of p-wave condensate where the magnetic field may be induced by a spin texture.

Since the effective models for the spin-triplet superconductor and the two-band superconductor appear as formally very similar, we refer a reader to the paper [2] for a detailed description of the knotted solitons in the model (13), whereas below we outline differences of the properties of solitons in these systems. One of the differences is that in the spin-triplet case the knot soliton may form as a nontrivial texture with no inhomogeneities in Cooper pair density. In particular, this means that there is no mass of the components of the vector $\vec{\mathbf{s}}$ whereas in the two-band case [2] the n_3 component of the unit vector $\vec{\mathbf{n}}$ is related to relative local Cooper pair densities and is indeed massive. In principle, in triplet case the components of the field $\vec{\mathbf{s}}$ may also acquire a mass if we take into account spin-orbit interaction which would give $\vec{\mathbf{s}}$ an energetically preferred direction. Here we assume spin-orbit interaction to be small compared with mass for the field $\vec{C}$. Under the assumption of a small spin-orbit interaction the characteristic size of a knotted soliton is determined by a competition of the second-order and fourth-order derivative terms for $\vec{\mathbf{s}}$, and thus it is of order of magnitude of magnetic field penetration length [14].

We stress that in a two-band superconductor [2], in the points where the unit vector $\vec{\mathbf{n}}$ is situated on the south or north poles of unit sphere S^2, the densities of Cooper pairs of flavor 1 or flavor 2 vanish correspondingly. Since a knot soliton is characterized by a Hopf invariant that means that the vector $\vec{\mathbf{n}}$ necessary hits poles of the unit sphere which necessarily results in zeros of Cooper pair densities. Also the contribution to the self-induced magnetic field associated with the term $\vec{\mathbf{n}} \cdot \partial_i \vec{\mathbf{n}} \times \partial_j \vec{\mathbf{n}}$ vanishes in these areas. In contrast, the knotted soliton with the same topology in a spin-triplet superconductor is a nontrivial configuration of spin texture, and it does not feature zeros of density of Cooper pairs. When spin-orbit interaction is small, we can choose, e.g., that the vector $\vec{\mathbf{s}}$ assumes at infinity the value corresponding to the north pole of the unit sphere and in the knot soliton core reaches the south pole. The configuration of the self-induced magnetic field associated with the contribution of $\vec{\mathbf{s}} \cdot \partial_i \vec{\mathbf{s}} \times \partial_j \vec{\mathbf{s}}$ in the spin-triplet case is the following: the magnetic field vanishes indeed at spatial infinity (north pole of the unit sphere) and also the magnetic field vanishes in the knotted soliton "core" [15]. In between the core and "knot boundary" the self-induced magnetic field associated with the contribution of $\vec{\mathbf{s}} \cdot \partial_i \vec{\mathbf{s}} \times \partial_j \vec{\mathbf{s}}$ has a helical geometry characterized by a corresponding Hopf invariant.

Thus the knot soliton in triplet superconductor has different structural features and different underlying physics than the knotted solitons in the two-gap superconductor, albeit the main feature that these defects have in common is the self-induced nontrivial configuration of magnetic field which amounts to a Faddeev term in the effective action and which stabilizes the size of the soliton. In other words, a shrinkage of a knotted soliton would result in an increasing energy of the self-induced magnetic field which gives this defect energetic stability (compare with [1–3]).

Let us also emphasize that the "polar" phase of triplet superconductors is not described by a version of the Faddeev model and does not allow formation of knotted solitons in contrast to the ferromagnetic phase. The properties of a neutral polar phase were investigated in [13]. This state appears in the case when $c_2 > 0$. The energy is minimized by $\langle \mathbf{F} \rangle = 0$. The spinor ζ in the ground state was calculated in [13]. In a charged counterpart of the system in the polar phase considered in [13], it can easily be seen that the supercurrent does not depend on the spin texture, and this formally results in the absence of a Faddeev term in the effective action.

In conclusion, we derived an exact equivalent presentation of the free energy functional for the ferromagnetic spin-triplet superconductor. The derived functional in dual gauge-invariant variables explicitly displays the physical degrees of freedom in the system. In particular, it allows us to conclude that the ferromagnetic spin-triplet superconductors allow formation of stable knotted solitons. This is in contrast to a neutral spin-1 condensate where the finite-length defects characterized by a nontrivial Hopf invariant are not energetically stable [9–12]. The amazing fact is that the derived model is very similar to the model considered in [2] despite being derived from a very different Ginzburg-Landau functional. It shows that this version of the Faddeev model is a rather generic model for various superconductors. The interesting problem is finding an experimental procedure to create and observe the knotted solitons. Indeed, in contrast to Abrikosov vortices these defects cannot be created by simply applying an external magnetic field. On the other hand, the situation is simplified by the key feature of these defects, which is that once they are formed they are stable, being protected against decay by an energy barrier. One may expect that, e.g., a rapid cooling of a system from above to below critical temperature in, e.g., applied random fields should indeed result in a formation of a certain density of topological defects. Apparently an ensemble of these defects, because of its complex structure, and very different nature comparing to Abrikosov vortices, should exhibit many unconventional phenomena and thus should be a very interesting object for experimental studies, especially since the triplet superconductivity has been established experimentally in several compounds (e.g., UPt_3 and Sr_2RuO_4 [16]). Finally, we especially stress that it was suggested in [3] that knotted solitons could play an important role in the infrared limit of QCD. The fact that similar defects should be present in p-wave and two-band superconductors and the macroscopic quantum origin of the topological defects in condensed matter systems which implies its rather direct observability means that the spin-triplet superconductors along with two-band superconductors may in some sense serve as "a testing laboratory" for the infrared limit of QCD [17].

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[14] The coupling to the field $\vec{C}$ also should affect a knot soliton: a nontrivial configuration of texture $\vec{\mathbf{s}}$ induces a magnetic field which in turn should induce screening Meissner current which amounts to a nontrivial configuration of $\vec{C}$. *However, the coupling to $\vec{C}$ could not kill the knot soliton.* Indeed, let us assume a knot is shrinking to a characteristic size smaller than the magnetic field penetration length λ, then we have a nontrivial configuration of $\vec{\mathbf{s}}$ which amounts to a self-induced magnetic field configuration proportional to $\vec{\mathbf{s}} \cdot \nabla_i \vec{\mathbf{s}} \times \nabla_j \vec{\mathbf{s}}$ which becomes more and more singular if the soliton is shrinking. The direction of this self-induced field also changes inside the knot since the configuration of $\vec{\mathbf{s}}$ is helical and is characterized by a Hopf invariant. On the other hand, the Meissner current $\vec{C}$ screens magnetic field in a superconductor over a characteristic length scale λ; thus, if a texture $\vec{\mathbf{s}}$ shrinks at characteristic sizes smaller than λ, we have a fast variable $\vec{\mathbf{s}}$ and a slow variable $\vec{C}$ over which we can average. Thus, at the length scales smaller than λ the model reduced to the model (1) with an arbitrary accuracy, and the knot solitons in the model (13) are stable. This can also be understood intuitively: at the length scales smaller than λ the system almost does not display Meissner effect, and thus the screening current cannot affect the highly nontrivial configuration of self-induced magnetic field proportional to $\vec{\mathbf{s}} \cdot \nabla_i \vec{\mathbf{s}} \times \nabla_j \vec{\mathbf{s}}$.

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