Quantifying Nonclassicality of One-Mode Gaussian States of the Radiation Field

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We define the degree of nonclassicality of a one-mode Gaussian state of the quantum electromagnetic field in terms of the Bures distance between the state and the set of all classical one-mode Gaussian states. We find the closest classical Gaussian state and the degree of nonclassicality using a recently established expression for the Uhlmann fidelity of two single-mode Gaussian states. The decrease of nonclassicality under thermal mapping is carefully analyzed. Along the same lines, we finally present the evolution of nonclassicality during linear amplification.

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The quantum nature of light manifests itself in special states of the quantum electromagnetic field, such as squeezed, antibunched, and sub-Poissonian states. Such states, whose correlation functions cannot be reproduced by any classical field, are called *nonclassical*. It is well known that, in addition to having been predicted by theory, nonclassical states have also been prepared in the laboratory.

By contrast, as pointed out by Glauber [1], there are states of the quantum radiation field for which all the normally ordered quantities have classical distributions. These states, which are now termed *classical,* have been characterized by Titulaer and Glauber [2] as possessing a well-behaved P representation of the density operator. We recall that a well-behaved P representation is either a non-negative regular function or a distribution no more singular than Dirac's δ. Cahill [3] and later on Hillery [4] proved that the only pure states that are classical are the coherent ones: all other classical states are mixtures.

It was first shown by Hillery [5] that a distance between a state and the set of all classical ones could serve as a measure for how much the distributions of observable quantities in the given state differ from classical distributions. Hillery employed the trace metric for one-mode states [5,6] and gave upper and lower bounds of this *nonclassical distance*. Recall that the Hilbert-Schmidt metric, quite extensively used in quantum optics [7,8], has recently been applied to the same purpose by Dodonov *et al.* [9]. However, these authors have chosen as reference sets of classical states the set of all coherent ones in the pure-state case and the set of all displaced thermal ones in the mixed-state case. We also refer to an important paper by Lee [10] who defines a *nonclassical depth* τ_m of any one-mode quantum state as follows: it is the minimum average number of photons added when a thermal field is superposed under the constraint that the resulting state be a classical one.

In this Letter, we adopt Hillery's distance-definition of nonclassicality. Nevertheless, we choose to express the ideal degree of nonclassicality Q of a state ρ in terms of its Bures distance [11] to the set C of all classical states as

$$Q(\rho) := \frac{1}{2} \min_{\rho' \in C} D_B^2(\rho, \rho'). \tag{1}$$

Recall that the Bures distance between two density operators ρ and σ acting on a Hilbert space $\mathcal{H}_A$ is related to the Uhlmann fidelity [12] of the two states:

$$D_B(\rho, \sigma) = [2 - 2\sqrt{\mathcal{F}(\rho, \sigma)}]^{1/2}. \tag{2}$$

Uhlmann introduced the function $\mathcal{F}(\rho, \sigma)$, now called *fidelity* [13], as the maximal quantum-mechanical transition probability between any purifications $|\Psi_\rho\rangle$ and $|\Psi_\sigma\rangle$ of the two states:

$$\mathcal{F}(\rho, \sigma) = \max |\langle \Psi_\rho | \Psi_\sigma \rangle|^2. \tag{3}$$

The pure states $|\Psi_\rho\rangle$ and $|\Psi_\sigma\rangle$ are elements of an extended Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ such that their reductions are the given states $\rho = \mathrm{Tr}_B(|\Psi_\rho\rangle\langle\Psi_\rho|)$ and $\sigma = \mathrm{Tr}_B(|\Psi_\sigma\rangle\langle\Psi_\sigma|)$. Uhlmann succeeded in obtaining the explicit expression of the fidelity,

$$\mathcal{F}(\rho, \sigma) = \{\mathrm{Tr}[(\sqrt{\rho}\, \sigma \sqrt{\rho})^{1/2}]\}^2. \tag{4}$$

If at least one of the states is pure, Eq. (4) reduces to the usual transition probability $\mathcal{F}(\rho, \sigma) = \mathrm{Tr}(\rho\sigma)$, which is operationally testable. Our option for the Bures distance is highly motivated by the description of the measuring process in quantum mechanics. Indeed, the Bures distance $D_B(\rho, \sigma)$ provides the best probabilistic distinguishability

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of the quantum states ρ and σ [14–16]: this means that the fidelity $\mathcal{F}(\rho, \sigma)$ equals the squared minimal overlap of the probability distributions $p_\rho(b) = \mathrm{Tr}(\rho E_b)$ and $p_\sigma(b) = \mathrm{Tr}(\sigma E_b)$ for the outcomes of any positive operator-valued measure (POVM) $\{E_b\}$,

$$\mathcal{F}(\rho, \sigma) = \left[\min_{\{E_b\}} \sum_b \sqrt{p_\rho(b)} \sqrt{p_\sigma(b)} \right]^2. \tag{5}$$

We deal with one-mode Gaussian states of the radiation field which are important both theoretically and experimentally. A Gaussian state is defined by a characteristic function (CF) of the form [17]

$$\chi(\lambda) = \exp\left[-\left(A + \frac{1}{2}\right)|\lambda|^2 - \frac{1}{2} B^* \lambda^2 - \frac{1}{2} B (\lambda^*)^2 + C^* \lambda - C \lambda^* \right], \qquad (A \geq 0), \tag{6}$$

and can be parametrized as a displaced squeezed thermal state (DSTS):

$$\rho = D(\alpha) S(r, \varphi) \rho_T S^\dagger(r, \varphi) D^\dagger(\alpha). \tag{7}$$

Here $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ is a Weyl displacement operator with the coherent amplitude α, $S(r, \varphi) = \exp\{\frac{1}{2} r [e^{i\varphi}(a^\dagger)^2 - e^{-i\varphi} a^2]\}$ is a Stoler squeeze operator with the squeeze factor r and squeeze angle φ, and

$$\rho_T = \frac{1}{\bar{n} + 1} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{\bar{n} + 1} \right)^n |n\rangle\langle n| \tag{8}$$

is the density operator of a thermal state with the mean occupancy $\bar{n}$. The DSTS parametrization is explicitly

$$A + \frac{1}{2} = \left(\bar{n} + \frac{1}{2}\right) \cosh(2r), \qquad B = -\left(\bar{n} + \frac{1}{2}\right) e^{i\varphi} \sinh(2r), \qquad C = \alpha. \tag{9}$$

Note that the broad class of Gaussian states contains pure states such as coherent and squeezed coherent ones and mixed states such as displaced thermal and squeezed thermal ones. Moreover, superposition of a thermal field on a Gaussian one yields a Gaussian mixed state of the field. Interest in the nonclassical properties of Gaussian states was recently renewed by the experimental realization of the teleportation of a one-mode coherent state [18]. For a possible similar experiment using a nonclassical state it became important to evaluate the extent to which the nonclassicality can survive imperfect teleportation [19].

The set C_0 of all classical Gaussian states is singled out by the condition that the Glauber-Sudarshan P representation be a well-behaved quasiprobability distribution. It was found [17] that for any classical Gaussian state the squeeze factor does not exceed a nonclassicality threshold r_c:

$$r \leq r_c := \frac{1}{2} \ln(2\bar{n} + 1). \tag{10}$$

Note that C_0 includes the two reference sets of classical states proposed by Dodonov *et al.* [9]. In what follows we shall call *classical* those unitary transformations in Hilbert space which map the coherent states into coherent states. Otherwise stated, a transformation is classical if and only if it preserves the minimum equal variances of the quadratures in each mode of the field, which are precisely those in the vacuum. We notice that the only one-mode classical transformations are the translations $D(\lambda)$ and the rotations $R(\theta) = \exp(-i\theta a^\dagger a)$.

In order to handle Eq. (1) in the case of a Gaussian state, either pure or mixed, we find it convenient to modify slightly Hillery's definition by restricting the set C to its subset C_0. Consequently, we evaluate the degree of nonclassicality

$$Q_0(\rho) := \frac{1}{2} \min_{\rho' \in C_0} D_B^2(\rho, \rho') \tag{11}$$

as an upper bound estimate of the ideal quantity (1). However, to be acceptable as a measure of nonclassicality, $Q_0(\rho)$ has to satisfy the following three requirements: ($Q1$) The degree of nonclassicality vanishes if and only if the state is classical; ($Q2$) Classical transformations preserve the degree of nonclassicality; ($Q3$) Nonclassicality does not increase under a POVM. In addition, we will show below that our definition (11) is appropriate in the sense that it fully agrees with the earlier result of Lee [10].

Our starting point in evaluating the amount of nonclassicality (11) is the explicit formula for the fidelity of two Gaussian states. It has recently been derived [20] by Twamley for squeezed thermal states (STS's), and by Scutaru and Paraoanu for DSTS's. We have used an expression in terms of the coefficients introduced in Eq. (6) [21]:

$$\mathcal{F}(\rho, \rho') = [(\Delta + \Lambda)^{1/2} - \Lambda^{1/2}]^{-1} \times \exp\left\{ -\frac{1}{\Delta} \left[(A + A' + 1)|C - C'|^2 + \frac{1}{2}(B + B')(C^* - C'^*)^2 + \frac{1}{2}(B^* + B'^*)(C - C')^2 \right] \right\} \tag{12}$$

with

$$\Delta := (A + A' + 1)^2 - |B + B'|^2, \tag{13}$$

$$\Lambda := 4\left[\left(A + \frac{1}{2}\right)^2 - |B|^2 - \frac{1}{4} \right] \left[\left(A' + \frac{1}{2}\right)^2 - |B'|^2 - \frac{1}{4} \right]. \tag{14}$$

If the DSTS ρ is classical ($r \leq r_c$), then the minimal value (11) is reached for $\rho' = \rho$, the unique state for which fidelity equals unity. Hence,

$$Q_0(\rho) = 0, \qquad (r \leq r_c), \tag{15}$$

as required by condition ($Q1$).

If, on the contrary, ρ is a nonclassical state ($r > r_c$), then maximization of the fidelity under the condition $r' \leq r'_c$ is achieved for a unique classical Gaussian state $\rho' = \tilde{\rho}$ having the parameters

$$\tilde{\alpha} = \alpha, \qquad \tilde{\varphi} = \varphi, \qquad \tilde{r} \equiv \tilde{r}_c := \frac{1}{2}\ln(2\tilde{\bar{n}} + 1), \tag{16}$$

and

$$\tilde{\bar{n}} = -\frac{1}{2} + \frac{1}{2}[1 + 2\sinh(2r_c)\exp(2r)]^{1/2}. \tag{17}$$

With these values we get the result

$$Q_0(\rho) = 1 - [\mathrm{sech}(r - r_c)]^{1/2}, \qquad (r > r_c). \tag{18}$$

Note that the parameters $\tilde{r}$, Eq. (16), and $\tilde{\bar{n}}$, Eq. (17), as well as the amount of nonclassicality given by Eqs. (15) and (18) are independent of the values α and φ. This means that the degree of nonclassicality $Q_0(\rho)$ is preserved by translations and rotations in phase space. Therefore, it meets the demand ($Q2$) of being invariant under all classical unitary transformations.

It follows from Eqs. (16) and (17) that, for mixed states ($\bar{n} > 0$), the closest classical state is mixed ($\tilde{\bar{n}} > 0$) and its squeeze factor satisfies the inequalities $r_c < \tilde{r} < r$. For pure states ($\bar{n} = 0$), we get $\tilde{\bar{n}} = 0$ and $\tilde{r} = 0$. A pure one-mode Gaussian state can always be parametrized as a displaced squeezed vacuum state (DSVS), $\rho = D(\alpha)S(r,\varphi)\,|0\rangle\langle 0|S^\dagger(r,\varphi)D^\dagger(\alpha)$, which has the degree of nonclassicality $Q_0(\rho) = 1 - (\mathrm{sech} r)^{1/2}$. The closest classical Gaussian state is in this case the coherent state having the same coherent amplitude $\tilde{\rho} = |\alpha\rangle\langle\alpha|$.

When expressed in terms of the parameters A, B, C, Eq. (9), the degree of nonclassicality becomes

$$Q_0(\rho) = 0, \qquad (A \geq |B|), \tag{19}$$

$$Q_0(\rho) = 1 - 2^{1/4}\frac{(A + 1/2 - |B|)^{1/4}}{(A + 1 - |B|)^{1/2}}, \qquad (A < |B|). \tag{20}$$

Equations (19) and (20) are the main result of this paper. They give the degree of nonclassicality of any Gaussian state (6). We show the importance of these formulas by studying the behavior of the nonclassicality of the Gaussian state ρ under a Gaussian noise mapping [1,22]

$$\Gamma_{\bar{m}}(\rho) := \frac{1}{\pi\bar{m}}\int d^2\beta \exp\left(-\frac{|\beta|^2}{\bar{m}}\right)D(\beta)\rho D^\dagger(\beta), \qquad (\bar{m} \geq 0), \tag{21}$$

where $\bar{m}$ is the mean number of added thermal photons. Since the thermal map (21) models a nonorthogonal POVM, we are now in a position to check the requirement ($Q3$). The only modification of the CF (6) under the mapping (21) is the addition of $\bar{m}$ to A, $A \rightarrow A' := A + \bar{m}$. By applying Eq. (20), it is easy to prove that the degree of nonclassicality $Q_0[\Gamma_{\bar{m}}(\rho)]$ of the Gaussian state $\Gamma_{\bar{m}}(\rho)$ *decreases* with the thermal noise $\bar{m}$, in accordance with condition ($Q3$). Relatedly, thermalization raises the nonclassicality threshold:

$$r_c(\bar{m}) = \frac{1}{2}\ln\frac{2\bar{n} + 1}{1 - 2\bar{m}}, \qquad \left(\bar{m} < \frac{1}{2}\right). \tag{22}$$

The degree of nonclassicality of the thermalized state (21) is explicitly

$$Q_0[\Gamma_{\bar{m}}(\rho)] = 0, \qquad [r \leq r_c(\bar{m})], \tag{23}$$

$$Q_0[\Gamma_{\bar{m}}(\rho)] = 1 - \left(\frac{2v}{v^2 + 1}\right)^{1/2}, \qquad [r > r_c(\bar{m})]. \tag{24}$$

In Eq. (24) we have used the notation

$$v^2 := 1 - (1 - 2\bar{m})(1 - \exp\{-2[r - r_c(\bar{m})]\}). \tag{25}$$

A laboratory implementation of the thermalization rule (21) is the phase-insensitive linear amplification of the field state [23,24]. We have used the corresponding master equation [24] to evaluate the time development of the degree of nonclassicality Q_0 of amplified DSTS's. Figure 1 shows that it decreases monotonically with the gain of the amplifier. As expected [24], the threshold gain G_c for the emergence of classicality is bounded by 2.

Lee's already-mentioned nonclassicality depth τ_m [10] of a nonclassical DSTS ρ is determined by the threshold condition

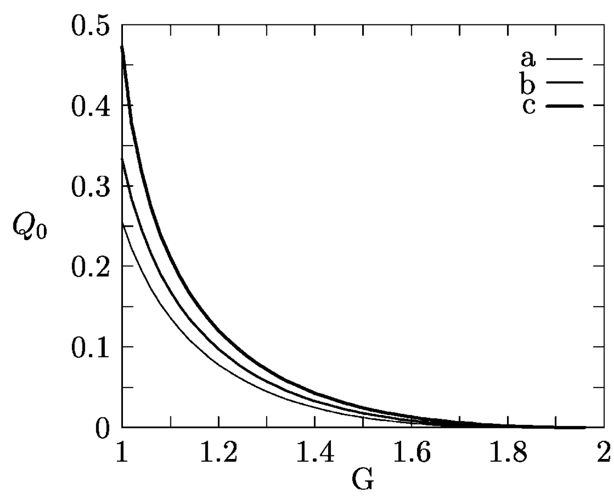

FIG. 1. Degree of nonclassicality versus gain in the linear amplification of DSTS's with the squeeze factor $r = 2$. The threshold gain is $G_c = 1.82$ for $\bar{n} = 2$ (plot a), $G_c = 1.88$ for $\bar{n} = 1$ (plot b), $G_c = 1.96$ for $\bar{n} = 0.05$ (plot c).

$$r_c(\tau_m) = r, \qquad \left(\tau_m < \frac{1}{2}\right). \qquad (26)$$

Taking into account Eqs. (10) and (22), insertion of Eq. (26) into Eq. (18) yields

$$Q_0(\rho) = 1 - \left[1 - \left(\frac{\tau_m}{1 - \tau_m}\right)^2\right]^{1/4}, \qquad (r > r_c). \qquad (27)$$

Accordingly, the amount of nonclassicality $Q_0(\rho)$ is a bijective function of the nonclassical depth τ_m: it increases from 0 to 1 when τ_m increases from 0 to 1/2. We also write the inverse function

$$\tau_m = \frac{1}{2}\left[1 - \left(\frac{[1 - Q_0(\rho)]^2}{1 + \{1 - [1 - Q_0(\rho)]^4\}^{1/2}}\right)^2\right], \qquad (r > r_c). \qquad (28)$$

Equations (27) and (28) therefore prove that the degree of nonclassicality $Q_0(\rho)$ and the nonclassical depth τ_m are *equivalent* measures of the nonclassicality of a single-mode Gaussian state. On the one hand, hence we can draw the conclusion that our definition (11) founded on the use of the reference set C_0 is adequate. On the other hand, the equivalence displayed by Eq. (27) is not at all trivial. Indeed, we have found that a quantity similar to $Q_0(\rho)$, but built with the Hilbert-Schmidt metric instead of the Bures metric, is not a function of the only variable τ_m. This favors the Bures distance in comparison with other distances in defining the amount of nonclassicality. However, one may ask why do we actually prefer $Q_0(\rho)$ to τ_m as a measure of nonclassicality, in spite of the more general character of the latter? One possible answer is that our Bures distance-definition of the one-mode nonclassicality can be extended to study the inseparability of two-mode states. In fact, inseparability is a stronger form of nonclassicality than that based on the P distribution. We have effectively applied the ideas sketched in this Letter to define and evaluate elsewhere an entanglement measure for an important class of bipartite mixed states of the radiation field, namely the two-mode STS's [25].

To sum up, we have succeeded in quantifying the nonclassicality of any single-mode Gaussian state. Our approach to solving this significant problem makes use of the Bures distance between Gaussian states, which has recently been evaluated. The suggestive result obtained in the DSTS parametrization, Eqs. (15) and (18), is applicable to any Gaussian state via Eqs. (19) and (20).

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