Noncommutative Quantum Mechanics from Noncommutative Quantum Field Theory

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We derive noncommutative multiparticle quantum mechanics from noncommutative quantum field theory in the nonrelativistic limit. Particles of opposite charges are found to have opposite noncommutativity. As a result, there is no noncommutative correction to the hydrogen atom spectrum at the tree level. We also comment on the obstacles to take noncommutative phenomenology seriously and propose a way to construct noncommutative SU(5) grand unified theory.

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1. Introduction.—Recently there has been a growing interest in noncommutative geometry as well as its phenomenological implications. This was motivated by the discovery in string theory that the low energy effective theory of D-brane in the background of Neveu-Schwarz–Neveu-Schwarz (NS-NS) B field lives on noncommutative space [1–6]. In the brane world scenario [7], our spacetime may be the world volume of a D-brane and thus may be noncommutative. In fact, apart from string theory, it has long been suggested that the spacetime may be noncommutative as a quantum effect of gravity, and it may provide a natural way to regularize quantum field theories [8,9].

In many proposals to test the hypothetical spacetime noncommutativity, one does not need the exact quantum field theory, but only its quantum mechanical approximation. Although noncommutative quantum mechanics (NCQM) has been extensively studied, we want to clarify a point that has not been emphasized before or was even misunderstood in some of these papers. The main point is that the noncommutativity θ^{ab} is not the same for all particles in NCQM. The noncommutativity of a particle should be opposite to (differ by a sign from) that of its antiparticle, and the noncommutativity of a charged particle should be opposite to any other particle of opposite charge. Our basic assumption is that NCQM should be viewed as an approximation of a noncommutative field theory (NCFT) in which all fields live on the same noncommutative space. The same viewpoint was taken in [10].

We always assume that the time coordinate t is commutative. Otherwise the formulation of quantum mechanics may require drastic modification [11].

Naively, to define a physical system on noncommutative space, we simply take the Lagrangian for ordinary space and replace all products by star products. For example, one tends to claim that the noncommutative Schrödinger equation for a hydrogen atom [12,13] is as follows [There is an ambiguity in the ordering of the last term. It could as well be $\psi * V$. However, replacing $V * \psi$ by $\psi * V$ is equivalent to replacing θ by $-\theta$. Without specifying θ, we can choose either case without loss of generality.]:

$$i\frac{\partial}{\partial t}\psi = -\frac{\nabla^2}{2m_e}\psi + V(x) * \psi, \tag{1}$$

where $V = -e^2/|x|$ is the electric potential of the proton, and the $*$ product is defined by

$$f(x) * g(x) \equiv e^{\frac{i}{2}\theta^{ab}\frac{\partial}{\partial x^a}\frac{\partial}{\partial x'^b}} f(x)g(x')|_{x'=x}. \tag{2}$$

Here x should be interpreted as the relative coordinate between the electron and the proton

$$x^a = x_e^a - x_p^a, \qquad a = 1,2,3. \tag{3}$$

This means that the commutation relation for x should be derived from those for x_e and x_p. Suppose

$$[x_e^a, x_e^b] = i\theta_e^{ab}, \qquad [x_p^a, x_p^b] = i\theta_p^{ab}, \qquad [x_e^a, x_p^b] = 0, \tag{4}$$

then

$$[x^a, x^b] = i(\theta_e^{ab} + \theta_p^{ab}). \tag{5}$$

We show below that we should take $\theta_e = -\theta_p$ and thus x is actually commutative.

If we assume that the proton has infinite mass and is localized at the origin as a delta function, we can interpret x as the coordinate of the electron. Then it would make sense to say that x is a coordinate on the noncommutative space. However, it is unnatural to assume an extreme localization of proton on a noncommutative space.

2. NCQM from NCFT.—Consider the NCFT of some charged particles and a U(1) gauge field. The action is of the form

$$S = \sum_\alpha S_\alpha + S_A. \tag{6}$$

S_α is the action for a charged particle. For instance, for a fermion in the fundamental representation of the gauge group, it is

$$S_\alpha = \int d^4x\, \bar{\psi}_\alpha * (i\not{D} + m_\alpha) * \psi_\alpha, \tag{7}$$

where m_α is the mass of the particle α and $D_\mu = \partial_\mu + A_\mu$. The action for the U(1) gauge field is

$$S_A = \int d^4x\, F_{\mu\nu} * F^{\mu\nu}, \tag{8}$$

where

$$F_{\mu\nu} = [D_\mu, D_\nu]_* . \tag{9}$$

On a noncommutative space, even the U(1) gauge group is non-Abelian. Therefore all fields must have the same charge: particles have charge $+1$ and antiparticles have charge -1.

In order to derive NCQM from NCFT, we repeat what we do for the commutative case. First, we collect those terms in the action involving A_μ,

$$\int d^4x\, (J^\mu * A_\mu) + S_A , \tag{10}$$

where $J = \sum_\alpha j_\alpha$, and the current density for the fermion in (7) is

$$j_\alpha^\mu = i\psi_\alpha * (\gamma^0\gamma^\mu)^T * \psi_\alpha^\dagger . \tag{11}$$

Now we can integrate out A_μ and find the effective interaction between the charged particles. In the weak coupling limit or weak field limit where we can ignore the self-interaction of A_μ, one finds the effective interaction

$$\begin{aligned} S_I &= \int d^4x\, d^4x'\, J^\mu(x) * G_{\mu\nu}(x,x') *' = J^\nu(x') \\ &= \int dt\, H_I , \end{aligned} \tag{12}$$

where G is the photon propagator in a certain gauge, and $*'$ means star product with respect to x'.

Decompose each field into positive and negative frequency modes

$$\psi = \int d^3k\, [b_{ks}(t) u_{ks} e^{ik_ix^i} + d_{ks}^\dagger(t) v_{ks} e^{-ik_ix^i}], \tag{13}$$

where b is the annihilation operator for the particle, $d^\dagger$ is the creation operator for its antiparticle, and the particle index α is suppressed. We ignore the spinor index s as it does not play any role in our problem. In the operator formulation, one can define the field operators

$$\hat\psi_+ \equiv \int d^3k\, b_k(t) e^{ik_ix^i} \tag{14}$$

for the particle α, and

$$\hat\psi_- \equiv \int d^3k\, d_k(t) e^{ik_ix^i} \tag{15}$$

for its antiparticle $\bar\alpha$. The quantum mechanical wave function for a two-particle state $|\xi\rangle$ in the NCFT is

$$\Psi_{(\alpha\epsilon_1)(\beta\epsilon_2)}(x_1,x_2) \equiv \langle 0|\hat\psi_{\alpha\epsilon_1}(x_1)\hat\psi_{\beta\epsilon_2}(x_2)|\xi\rangle . \tag{16}$$

Here x_1, x_2 are viewed as commutative coordinates in the star product representation. Thus the coordinates for different particles in the wave function Ψ always commute with one another by definition. Similarly, one can define the wave function for a state of an arbitrary number of particles and antiparticles.

The Schrödinger equation is a result of the fact that ψ_α satisfies its equation of motion, which can be written as

$$i\dot\psi(x) = [H, \psi(x)] \tag{17}$$

in terms of the Hamiltonian H. For the effective action, $H = H_0 - H_\mathrm{I}$, where H_0 is the kinetic term and H_I is given by (12). Thus, for example,

$$i\frac{\partial}{\partial t}\Psi_{AB} = \langle = 0|[H, \hat\psi_A\hat\psi_B]|\xi\rangle , \tag{18}$$

where $A = (\alpha\epsilon_1)$ and $B = (\beta\epsilon_2)$.

Straightforward derivation shows that, in the nonrelativistic approximation where the interaction is dominated by the Coulomb potential, the Schrödinger equation is given by

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi_{AB}(x_1,x_2) &= \left(-\frac{\nabla_1^2}{2m_\alpha} - \frac{\nabla_2^2}{2m_\beta} + V(x_1,x_2)\right) \\ &\quad \times *_{\epsilon_1\epsilon_2}\Psi_{AB}(x_1,x_2) , \end{aligned} \tag{19}$$

where $*_{\epsilon_1\epsilon_2}$ is defined by

$$\begin{aligned} f(x_e,x_p) *_{\epsilon_1\epsilon_2} g(x_e,x_p) &\equiv e^{\frac{i}{2}\theta^{ab}(\epsilon_1\frac{\partial}{\partial x_e^a}\frac{\partial}{\partial x_e'^b} + \epsilon_2\frac{\partial}{\partial x_p^a}\frac{\partial}{\partial x_p'^b})} \\ &\quad \times f(x_e,x_p)g(x_e',x_p')|_{x=x'} , \end{aligned} \tag{20}$$

and V is given by $V(x_1,x_2) = -\frac{e^2}{|x_1-x_2|}$.

While the above prescription applies to generic interactions, for our special case of a gauge field, the result above (19) can be easily obtained by demanding gauge symmetry. For a field in the fundamental representation,

$$\hat\psi_+ \to \mathrm{U} * \hat\psi_+, \qquad \hat\psi_- \to \hat\psi_- * U^\dagger \tag{21}$$

under a gauge transformation. This implies that the covariant derivative must act on the wave function Ψ from the left for particles and from the right for antiparticles. Since the electric potential V is just the time component of the gauge potential A_μ, we immediately reach the same conclusion as in (19).

In the context of string theory, for an open string ending on a D-brane, the two end points appear as opposite charges to the D-brane gauge field. In a B field background, the two end points also observe opposite noncommutativity [3]. It was first argued in [3] that the NCFT on a single noncommutative space automatically takes care of this effect. In this section we provided a rigorous derivation.

3. Separation of Variables.—To solve the Schrödinger equation for multiparticle wave functions, we use the technique of separation of variables. For the hydrogen atom, the Schrödinger equation is

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi(x_e,x_p) &= \left(-\frac{\nabla_e^2}{2m_e} - \frac{\nabla_p^2}{2m_p} + V(x_e,x_p)\right) \\ &\quad \times *_{-+}\Psi(x_e,x_p) , \end{aligned} \tag{22}$$

where we choose the convention that the noncommutativity parameter θ $(-\theta)$ is associated with positive (negative) charges.

Since the kinetic term is not modified, we take the ansatz

$$\Psi(x_e, x_p) = \Phi(X)\psi(x), \tag{23}$$

where $X = \frac{m_e x_e + m_p x_p}{m_e + m_p}$ is the center of mass (c.m.) coordinate, and $x = x_e - x_p$ is the relative coordinate. The noncommutativity for these coordinates is given by

$$[X^i, X^j]_{*-+} = i\,\frac{m_p - m_e}{m_p + m_e}\,\theta^{ij} \equiv i\theta^{ij}_{pp}, \tag{24}$$

$$[x^i, x^j]_{*-+} = 0, \qquad [x^i, X^j]_{*-+} = i\theta^{ij} \equiv i\theta^{ij}_{ep}. \tag{25}$$

The kinetic term can be rewritten as

$$\frac{\nabla_e^2}{2m_e} + \frac{\nabla_p^2}{2m_p} = \frac{\nabla_X^2}{2M} + \frac{\nabla_x^2}{2m}, \tag{26}$$

where $M = m_e + m_p$ is the total mass and $m = \frac{m_e m_p}{m_e + m_p}$ is the reduced mass.

For the Fourier mode of X,

$$\Psi(X) = e^{-iEt + iK_i X^i}\psi(x), \tag{27}$$

(22) is reduced to

$$\left(E - \frac{K^2}{2M}\right)\psi(x) = \left[-\frac{\nabla_x^2}{2m} + V\left(x - \frac{1}{2}\,\theta_{ep}K\right)\right] \times \psi(x). \tag{28}$$

Note that translational invariance implies that V can depend only on the relative coordinate x. Let $\psi(x) = \psi'(x - \frac{1}{2}\theta_{ep}K)$. Since (28) contains no star product, it is exactly the same equation for classical space in terms of ψ'. Unless we include self-interactions of the gauge field, the whole spectrum is exactly the same as the commutative case. The shift in the relative coordinate is easy to understand from the D-brane picture, where space noncommutativity results from the background B field.

Therefore, for example, the noncommutative correction to Lamb shift should be much smaller than the one given in [12]. There is no correction at tree level. The lowest order contribution of θ comes from the one-loop diagrams and is negligible.

4. Generalization.—Suppose that there are m particles. Let the charges of particle α $(\alpha = 1, \dots, m)$ be q^α, where $q^\alpha = 1, 0, -1$. If $q^\alpha = 1(-1)$, it means that the field operator for particle α, which was denoted as $\hat\psi_+$ $(\hat\psi_-)$ before, transforms from the left (right) as in (21).

If we repeat the derivation in the previous sections, the Schrödinger equation for N particles is

$$i\,\frac{\partial}{\partial t}\,\Psi(x_1, \dots, x_N) = -\sum_{\alpha=1}^{N} \frac{\nabla_i^2}{2m_i}\,\Psi + \frac{1}{2}\sum_{\alpha\neq\beta} q_\alpha q_\beta V(x_\alpha, x_\beta) \times *_{q_\alpha q_\beta}\Psi, \tag{29}$$

where V is the (00) component of the Green's function for the gauge field A_μ.

The c.m. coordinates X^μ of the system satisfies

$$[X^\mu, X^\nu]_{*\{q_i\}} = i\,\frac{\sum_{\alpha=1}^N q_i^\alpha m_\alpha^2}{\sum_\beta m_\beta^2}\,\theta^{\mu\nu}. \tag{30}$$

It is easy to see that the magnitude of the noncommutativity is never larger than $|\theta|$.

A composite particle is a system of N particles which has a bound state with a small spatial extension. The c.m. coordinates of the system will be taken as the coordinates of the composite particle. If the size of the composite particle is larger than $\sqrt{\theta}$, it is meaningless to talk about its noncommutativity. In the case of a hydrogen atom, the relative coordinate x is commutative; thus its size can be arbitrarily small. On the other hand, if some relative coordinates for the constituents of the composite particle are noncommutative, which is always the case as long as there are three or more charged constituent particles, the size of the composite particle must be larger than the order of $\sqrt{\theta}$, and hence the noncommutativity of the composite particle can be neglected for most purposes.

5. Discussion.—On noncommutative (NC) space, charges are always quantized, even for the U(1) gauge field. However, in the standard model, there are particles of electric charges $1/3, 2/3$, etc. It implies that the electromagnetic interaction cannot be a NC U(1) gauge theory. Similarly, the U(1) gauge group for hypercharges cannot be noncommutative, either [14]. In the SU(5) grand unified theory (GUT), on the other hand, all charges are already quantized. There are fractional hypercharges only because the U(1) group is embedded in SU(5) with a generator $T = \mathrm{diag}(1/3, 1/3, 1/3, -1/2, -1/2)$. But there are other problems for NCGUT. The first problem is to define NC SU(5) gauge symmetry. In general, it is straightforward to construct NC U(N) gauge theory, but difficult to have any other gauge group [15–17].

A possible resolution of this problem [18] is to define NC SU(N) gauge symmetry as the image of the classical SU(N) via the Seiberg-Witten (SW) map [6]

$$\hat A = \hat A(A), \tag{31}$$

where quantities without (with) hats are commutative (noncommutative) fields. It is consistent with gauge transformations to restrict A_μ to the Lie algebra of SU(N). The same idea can be used to define the noncommutative version of any classical group [18].

It is also possible to define NC SU(5) theory directly in terms of the noncommutative variable $\hat A$ without mentioning the commutative A. We can simply take the NC U(N) gauge field $\hat A$ and impose the following constraint:

$$C_{\mu\nu}(k) \equiv \mathrm{Tr} F_{\mu\nu}(\hat A)(k) = 0, \tag{32}$$

where $F_{\mu\nu}(\hat A)$ is the inverse SW map. (An exact expression for the inverse SW map was given in [19].) This implies that the U(1) part of A_μ can be gauged away, and the result is equivalent to the approach of [18].

It is interesting to note that another constraint with a much simpler expression

$$C'_{\mu\nu}(k) \equiv \mathrm{Tr} \int d^4x\, \hat{F}_{\mu\nu}(x) * e_*^{ik_\mu(x^\mu + i\theta^{\mu\nu}\hat{A}_\nu)} = 0 \tag{33}$$

is also gauge invariant and has the same classical limit $\mathrm{Tr}F_{\mu\nu} = 0$. At this moment we do not know if these two constraints are exactly the same.

Recently, a similar idea was proposed independently in [20], where the constraint was imposed on $\hat{A}$ instead. Another constraint on gauge transformations $\hat{\mathrm{U}}$ has to be imposed simultaneously for consistency [20]. It would be of interest to know if all such constraints are equivalent under field redefinitions.

Another problem about NCGUT is that there are matter fields in the antisymmetric representation of SU(5). Although it is straightforward to define matter fields in the fundamental and adjoint representations, in general it is hard to introduce other representations [15,16].

This problem can also be solved by using the SW map. For any D dimensional representation of SU(5), we consider the SW map for NC U(D) gauge symmetry. For any classical gauge transformation $\mathrm{U} \in \mathrm{U}(D)$, the SW map provides a NC U(D) transformation $\hat{\mathrm{U}}(\mathrm{U})$. Since SU(5) can be embedded in U(D) according to its D dimensional representation, we can define NC SU(5) transformations for a fundamental representation of U(D) by

$$\hat{\phi}_a \to \hat{\mathrm{U}}_{ab}(\mathrm{U})\hat{\phi}_b, \qquad a, b = 1, 2, \ldots, D, \tag{34}$$

where U is a classical SU(5) gauge transformation. Thus $\hat{\phi}$ can be viewed as a D dimensional representation of NC SU(5).

For the ten dimensional representation of SU(5), one usually defines it as an antisymmetric tensor $\phi_{ij} = -\phi_{ji}$ $(i, j = 1, 2, \ldots, 5)$ which transforms as $\phi \to \mathrm{U}\phi\mathrm{U}^\dagger$. However, the tensor will not be antisymmetric after a generic NC gauge transformation. In the above we avoided this problem by defining this representation directly as a ten component column without any constraint.

Similarly, it is consistent with classical SU(5) gauge transformations to restrict the classical U(D) gauge potential $A^{(D)}$ to the su(5) Lie algebra embedded in u(D). Its image under the SW map can be viewed as the NC gauge potential in the D dimensional representation of NC SU(5). The covariant derivative of a matter field in this representation is

$$\hat{D}_\mu \hat{\phi} = [\partial_\mu + \hat{A}_\mu^{(D)}(A)]\hat{\phi}, \tag{35}$$

where A is the commutative SU(5) gauge potential. Obviously, this construction also works for other gauge groups.

Finally, due to the UV-IR mixing, the UV divergences of NC quantum field theories result in new IR poles nonperturbative in θ [21,22]. For a comprehensive discussion on this problem, see [23]. In order to give a reliable, consistent description of NC electromagnetic interactions, or any other low energy phenomena on NC space, it is necessary to properly address all these problems. We leave these issues for future study.

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