# Conformal Field Theory Interpretation of Black Hole Quasinormal Modes

Danny Birmingham,[1] Ivo Sachs,[2] and Sergey N. Solodukhin[2]

[1]*Department of Mathematical Physics, University College Dublin, Belfield, Dublin 4, Ireland*

[2]*Theoretische Physik, Ludwig-Maximilians Universität, Theresienstrasse 37, D-80333, München, Germany*

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We obtain exact expressions for the quasinormal modes of various spin for the Bañados-Teitelboim-Zanelli black hole. These modes determine the relaxation time of black hole perturbations. Exact agreement is found between the quasinormal frequencies and the location of the poles of the retarded correlation function of the corresponding perturbations in the dual conformal field theory. This then provides a new quantitative test of the anti–de Sitter/conformal field theory correspondence.

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The problem of how a perturbed thermodynamical system returns to equilibrium is an important issue in statistical mechanics and finite temperature field theory [1]. For a small perturbation, this process is described by linear response theory [1,2]. The relaxation process is then completely determined by the poles, in the momentum representation, of the retarded correlation function of the perturbation. On the other hand, black holes also constitute a thermodynamical system. At equilibrium, the various thermodynamical quantities, such as the temperature and the entropy, are determined in terms of the mass, charge, and angular momentum of the black hole. The decay of small perturbations of a black hole at equilibrium are described by the so-called quasinormal modes [3]. For asymptotically flat black hole space-times, quasinormal modes are analyzed by solving the wave equation for matter or gravitational perturbations, subject to the condition that the flux at the horizon is ingoing, with outgoing flux at asymptotic infinity. The wave equation, subject to these boundary conditions, admits only a discrete set of solutions with complex frequencies. The imaginary part of these quasinormal frequencies then determines the decay time of small perturbations or, equivalently, the relaxation of the system back to thermal equilibrium.

On another front, over the past few years increasing evidence has accumulated which shows that there is a correspondence between gravity and quantum field theory in flat space-time (for a review, see [4]). In particular, this duality has led to important progress in our understanding of the microscopic physics of a class of near-extremal black holes. The purpose of this Letter is to analyze whether such a correspondence exists between quasinormal modes in anti–de Sitter (AdS) black holes and linear response theory in scale invariant finite temperature field theory. A correspondence between quasinormal modes and the decay of perturbations in the dual conformal field theory (CFT) was first suggested in [5]. The analysis of [5] is based on the numerical computation of quasinormal modes for AdS-Schwarzschild black holes in four, five, and seven dimensions. Further numerical computations of quasinormal modes in asymptotically AdS space-times have been presented in [6–11]. For related discussions in the context of black hole formation, see [12]. Qualitative agreement was found with the results expected from the conformal field theory side. However, a quantitative test of such correspondence between quasinormal modes and the linear response of the dual conformal field theory is lacking so far. In this paper, we consider the $(2+1)$-dimensional AdS black hole [13], and show that there is a precise quantitative agreement between its quasinormal frequencies and the location of the poles of the retarded correlation function describing the linear response on the conformal field theory side. Both computations are performed analytically. As a result, we can identify not just the lowest, but the complete (infinite) set of frequencies on both sides of the AdS/CFT correspondence. In spite of its simplicity, this model plays an important role also for black holes in higher dimensions whose near-horizon geometry is $AdS_{2+1}$ (see, e.g., [14] for a review and references).

The metric of the Bañados-Teitelboim-Zanelli (BTZ) black hole is given by

$$ds^2 = -\sinh^2\mu(r_+dt - r_-d\phi)^2 + d\mu^2 + \cosh^2\mu(-r_-dt + r_+d\phi)^2. \quad (1)$$

The angular coordinate $\phi$ has period $2\pi$, and the radii of the inner and outer horizons are denoted by $r_-$ and $r_+$, respectively. We have also set to unity the radius of the anti–de Sitter space, $\ell \equiv 1$. The dual conformal field theory on the boundary is $(1+1)$-dimensional, the conformal symmetry being generated by two copies of the Virasoro algebra acting separately on left- and right-moving sectors. Consequently, the conformal field theory splits into two independent sectors at thermal equilibrium with temperatures

$$T_L = (r_+ - r_-)/2\pi, \qquad T_R = (r_+ + r_-)/2\pi. \quad (2)$$

According to the $AdS_3/CFT_2$ correspondence, to each field of spin $s$ propagating in $AdS_3$ there corresponds an

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operator $\mathcal{O}$ in the dual conformal field theory characterized by conformal weights $(h_L, h_R)$ with [4]

$$h_R + h_L = \Delta, \qquad h_R - h_L = \pm s, \tag{3}$$

and $\Delta$ is determined in terms of the mass $m$ of the field. In particular, we have

$$\Delta = 1 + \sqrt{1 + m^2}, \tag{4}$$

for scalar fields, and

$$\Delta = 1 + |m|, \tag{5}$$

for both fermionic and vector fields. For a small perturbation, the manner in which the field theory relaxes back to thermal equilibrium can then be analyzed within linear response theory [2]. One expects that at late times the perturbed system will approach equilibrium exponentially with a characteristic time scale. This time scale is inversely proportional to the imaginary part of the poles, in momentum space, of the correlation function of the perturbation operator $\mathcal{O}$. In this case, according to our proposal, the relevant correlation function is the retarded real time correlation function

$$\begin{aligned}\mathcal{D}^{\rm ret}(x,x') &= i\theta(t - t')\langle[\mathcal{O}(x), \mathcal{O}(x')]\rangle_T \\ &= i\theta(t - t')\bar{\mathcal{D}}(x,x'),\end{aligned} \tag{6}$$

where $\bar{\mathcal{D}}(x,x') = \mathcal{D}_+(x,x') - \mathcal{D}_-(x,x')$ is the commutator evaluated in the equilibrium canonical ensemble. For a conformal field theory at zero temperature, the 2-point correlation functions can be determined, up to a normalization, from conformal invariance. At finite temperature $T$, one has to take into account the infinite sum over images to render the correlation function periodic in imaginary time, with period $1/T$. The result of this summation in two dimensions was determined in [15], and depends only on the conformal dimensions $(h_L, h_R)$ of the perturbation operator. We have $(x^\pm = t \pm \sigma)$,

$$\mathcal{D}_+(x) = \frac{(\pi T_R)^{2h_R}}{\sinh^{2h_R}(\pi T_R x^- - i\epsilon)} \frac{(\pi T_L)^{2h_L}}{\sinh^{2h_L}(\pi T_L x^+ - i\epsilon)},$$

and a similar expression for $\mathcal{D}_-(x)$ with $\epsilon \to -\epsilon$. In order to determine the location of the poles, we need to compute the Fourier transform of (6). This is complicated by the presence of the $\theta$ function. We can, however, determine the location of the poles indirectly. For this we first consider the Fourier transform of the commutator $\bar{\mathcal{D}}(x)$. This integral can be evaluated using contour techniques, leading to [16]

$$\begin{aligned}\bar{\mathcal{D}}(k_+, k_-) \propto\ & \Gamma\left(h_L + i\frac{p_+}{2\pi T_L}\right)\Gamma\left(h_R + i\frac{p_-}{2\pi T_R}\right) \\ & \times \Gamma\left(h_L - i\frac{p_+}{2\pi T_L}\right)\Gamma\left(h_R - i\frac{p_-}{2\pi T_R}\right),\end{aligned} \tag{7}$$

where $p_\pm = \frac{1}{2}(\omega \mp k)$. This function has poles in both the upper and lower halves of the $\omega$ plane. The poles lying in the lower half-plane are the same as the poles of the retarded correlation function (6). Restricting the poles of (7) to the lower half-plane, we find two sets of poles

$$\begin{aligned}\omega_L &= k - 4\pi i T_L(n + h_L), \\ \omega_R &= -k - 4\pi i T_R(n + h_R).\end{aligned} \tag{8}$$

Here, and in the following, $n$ takes the integer values $(n = 0, 1, 2, \ldots)$. This set of poles characterizes the decay of the perturbation on the CFT side, and coincides precisely with the quasinormal frequencies of the BTZ black hole, as we shall now show for fields of various spin.

*Scalar perturbation* $(s = 0)$.—Scalar perturbations are described by the wave equation

$$(\nabla^2 - m^2)\Phi = 0. \tag{9}$$

We use the ansatz

$$\Phi = e^{-i(k_+x^+ + k_-x^-)}R(\mu), \tag{10}$$

where $x^+ = r_+t - r_-\phi$, $x^- = r_+\phi - r_-t$, and

$$\begin{aligned}(k_+ + k_-)(r_+ - r_-) &= \omega - k, \\ (k_+ - k_-)(r_+ + r_-) &= \omega + k.\end{aligned} \tag{11}$$

Here, $\omega$ and $k$ are the energy and angular momentum of the perturbation. By changing variables to $z = \tanh^2\mu$, we end up with the hypergeometric equation

$$\begin{aligned}z(1 - z)\frac{d^2R}{dz^2} + (1 - z)\frac{dR}{dz} + \\ \left[\frac{k_+^2}{4z} - \frac{k_-^2}{4} - \frac{m^2}{4(1 - z)}\right]R = 0.\end{aligned} \tag{12}$$

The solution which is ingoing at the horizon is given by

$$R(z) = z^\alpha(1 - z)^{\beta_s}F(a_s, b_s, c_s, z), \tag{13}$$

where $\alpha = -\frac{ik_+}{2}$, $\beta_s = \frac{1}{2}(1 - \sqrt{1 + m^2})$, and

$$\begin{aligned}a_s = \frac{(k_+ - k_-)}{2i} + \beta_s, \qquad b_s = \frac{(k_+ + k_-)}{2i} + \beta_s, \\ c_s = 1 + 2\alpha.\end{aligned} \tag{14}$$

The quasinormal modes for the scalar perturbations were found in [17] by imposing the vanishing Dirichlet condition at infinity. Here, we reevaluate these modes using the condition that the flux given by

$$\mathcal{F} = \sqrt{g}\,\frac{1}{2i}(R^*\partial_\mu R - R\partial_\mu R^*) \tag{15}$$

vanishes at asymptotic infinity. For $m^2 > 0$, the asymptotic flux has a set of divergent terms, with the leading term of order $(1 - z)^{2\beta_s}$. Each of these terms is proportional to [18]

$$\left|\frac{\Gamma(c_s)\Gamma(c_s - a_s - b_s)}{\Gamma(c_s - a_s)\Gamma(c_s - b_s)}\right|^2. \tag{16}$$

Thus, the asymptotic flux vanishes if $c_s - a_s = -n$, or

$c_s - b_s = -n$, i.e.,

$$\frac{i}{2}(k_+ \pm k_-) = n + \frac{1}{2}(1 + \sqrt{1 + m^2}). \tag{17}$$

Using (11), one sees that these are the quasinormal modes found in [17]. For the scalar bulk field, we have $h_L = h_R = \frac{1}{2}(1 + \sqrt{1 + m^2})$. Thus, we observe that (17) exactly reproduces (8).

In AdS space-time, a negative mass squared for a scalar field is consistent, as long as $-1 < m^2 < 0$. A detailed analysis shows that in this case there is a second set of modes with $a_s = -n$, or $b_s = -n$, that is $h_L = h_R = \frac{1}{2}(1 - \sqrt{1 + m^2})$. This is in fact expected since, for $-1 < m^2 < 0$, there are two sets of dual operators with $\Delta_+ = 1 + \sqrt{1 + m^2}$ and $\Delta_- = 1 - \sqrt{1 + m^2}$ [4]. The second set of quasinormal frequencies in this range then matches exactly the dual operators with $\Delta = \Delta_-$. We note in passing that the Dirichlet boundary condition suggested in [5] leads to the same quasinormal modes for $m^2 > 0$ but does not lead to any quasinormal modes for $m^2 < 0$.

*Fermion perturbation* ($s = 1/2$).—In [9], the quasinormal fermionic perturbation in the BTZ background has been analyzed numerically. However, as expected, it is possible to find analytic solutions in this simple case [19]. We begin with the Dirac equation

$$(\not{D} + m)\Psi = 0, \qquad \Psi = e^{-i(k_+x^+ + k_-x^-)}\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}. \tag{18}$$

Following [19], we make the substitutions

$$\psi_1 \pm \psi_2 = \sqrt{\frac{\cosh\mu \pm \sinh\mu}{\cosh\mu \sinh\mu}}(\psi_1' \pm \psi_2') \tag{19}$$

to obtain

$$2(1 - z)z^{1/2}\partial_z\psi_1' + i(k_+z^{-1/2} + k_-z^{1/2})\psi_1' = -[i(k_+ + k_-) + m + \tfrac{1}{2}]\psi_2' \tag{20}$$

and a similar equation, where $\psi_1'$, $\psi_2'$, and $k_\pm$ and $-k_\pm$ are interchanged. The solutions of these equations with ingoing flux at the horizon are given by

$$\psi_1' = z^\alpha(1 - z)^{\beta_f}F(a_f, b_f, c_f, z), \qquad \psi_2' = \left(\frac{a_f - c_f}{c_f}\right)z^{\alpha+1/2}(1 - z)^{\beta_f}F(a_f, b_f + 1, c_f + 1, z), \tag{21}$$

where $\alpha = -\frac{ik_+}{2}$, $\beta_f = -\frac{1}{2}(m + \frac{1}{2})$, $c_f = \frac{1}{2} + 2\alpha$, and

$$a_f = \frac{k_+ - k_-}{2i} + \beta_f + \frac{1}{2}, \qquad b_f = \frac{k_+ + k_-}{2i} + \beta_f. \tag{22}$$

The asymptotic form ($z \to 1$) of $\psi_1$ and $\psi_2$ can now be determined explicitly. In analogy with the scalar perturbations, we then impose the condition that the flux [19]

$$\mathcal{F} = \sqrt{g}\,\bar{\Psi}\gamma_\mu\Psi \simeq (1 - z)^{-1}(|\psi_1|^2 - |\psi_2|^2) \tag{23}$$

vanishes at infinity. The resulting quasinormal modes are then obtained as follows. For $m > 0$, the leading divergent term in the asymptotic flux is of order $(1 - z)^{2\beta+1}$. Vanishing flux then requires that the coefficient

$$\frac{\Gamma(c_f)\Gamma(c_f - a_f - b_f)}{\Gamma(c_f - a_f)\Gamma(c_f - b_f)} \tag{24}$$

vanishes, that is,

$$\frac{i}{2}(k_+ + k_-) = n + \frac{1}{4} + \frac{m}{2} \quad \text{or} \quad \frac{i}{2}(k_+ - k_-) = n + \frac{3}{4} + \frac{m}{2}. \tag{25}$$

The above conditions imply that all coefficients of the subleading, asymptotically nonvanishing, contributions to the flux also vanish. Thus, we have precise agreement with (8), where the left and right conformal weights are given by $h_L = \frac{1}{4} + \frac{1}{2}m$ and $h_R = \frac{3}{4} + \frac{1}{2}m$. For $m < 0$, one obtains a similar result with $h_L = \frac{3}{4} - \frac{1}{2}m$ and $h_R = \frac{1}{4} - \frac{1}{2}m$. Note again that imposing Dirichlet boundary conditions for $\psi_1$ and $\psi_2$ at infinity would lead to the absence of quasinormal modes for $-1 < m < 1$. We can think of no physical reason for the absence of quasinormal modes in this range of masses. Thus, we take this as another motivation for imposing vanishing flux at infinity, rather that Dirichlet conditions for asymptotically AdS space-times. One should also note that for positive mass the spinor perturbation is asymptotically left handed, whereas it is right handed for negative mass (see also [4]).

*Vector perturbation* ($s = 1$).—The massive Maxwell field in $AdS_3$ is described by the first order equation

$$\epsilon_\lambda{}^{\alpha\beta}\partial_\alpha A_\beta = -mA_\lambda. \tag{26}$$

Let

$$A_i = e^{-i(k_+x^+ + k_-x^-)}A_i(\mu), \tag{27}$$

where $A_{1,2} = A_+ \pm A_-$. Then, after changing variables as before, we recover the scalar equation (12) for $A_1$ and $A_2$ [19], where the scalar mass squared is replaced by $m^2 + 2\varepsilon_i m$, with $\varepsilon_1 = -\varepsilon_2 = 1$. As in the scalar case, the solutions with ingoing flux at the horizon are then given by

$$A_1 = e_1z^\alpha(1 - z)^{\beta_v+1}F(a_v + 1, b_v + 1, c_v, z), \qquad A_2 = e_2z^\alpha(1 - z)^{\beta_v}F(a_v, b_v, c_v, z), \tag{28}$$

where $\alpha = -\frac{ik_+}{2}$, $\beta_v = \frac{m}{2}$, $c_v = 1 + 2\alpha$, and

$$a_v = \frac{(k_+ - k_-)}{2i} + \beta_v, \qquad b_v = \frac{(k_+ + k_-)}{2i} + \beta_v . \tag{29}$$

Note that the two "scalar modes" in (27) are not independent. The first order equation (26) relates the two coefficients $e_1$ and $e_2$ by

$$\frac{e_2}{e_1} = \frac{i(k_+ - k_-) + m}{i(k_+ + k_-) - m} = \frac{c_v - b_v - 1}{b_v}. \tag{30}$$

The remaining component $A_\mu$ is related to $A_\pm$ by

$$A_\mu = \frac{1}{m \cosh\mu \sinh\mu} \partial_{[+}A_{-]} . \tag{31}$$

For a real vector field, the particle flux is not defined. One way to avoid this difficulty is to consider a complex vector field. Alternatively, one can consider the energy flux divided by the redshifted frequency [16]. Both approaches lead to the same conditions, namely, that $A_1$ and $A_2$ vanish at infinity. Thus, we impose the Dirichlet boundary condition for $A_{1,2}$. Using (30), one finds that the leading asymptotic behavior ($z \rightarrow 1$) of the solutions (28) for positive $m$ is given by

$$\begin{aligned} A_1 &\simeq (1 - z)^{-m/2} \frac{\Gamma(c_v)\Gamma(a_v + b_v - c_v + 2)}{\Gamma(a_v + 1)\Gamma(b_v)}, \\ A_2 &\simeq (1 - z)^{1-m/2}(c_v - b_v - 1) \frac{\Gamma(c_v)\Gamma(a_v + b_v - c_v)}{\Gamma(a_v)\Gamma(b_v)} . \end{aligned} \tag{32}$$

By imposing the vanishing Dirichlet condition at infinity for the components $A_1$ and $A_2$, we find the quasinormal modes $a_v + 1 = -n$ or $b_v = -n$, i.e.,

$$\frac{i}{2}(k_+ - k_-) = n + 1 + \frac{m}{2} \quad \text{or} \quad \frac{i}{2}(k_+ + k_-) = n + \frac{m}{2}. \tag{33}$$

Now, for spin $s = 1$, the conformal weights are either $|m|/2$ or $1 + |m|/2$. Thus, we again find agreement with (8), where the left and right conformal weights are given by $h_L = \frac{1}{2}m$ and $h_R = 1 + \frac{1}{2}m$. For negative $m$, the situation is analogous to the fermionic perturbations. One finds the same conditions with $h_L = 1 - \frac{1}{2}m$ and $h_R = -\frac{1}{2}m$.

In conclusion, we have shown that there is quantitative agreement between the quasinormal frequencies of the BTZ black hole and the poles of the retarded correlation function of the corresponding perturbations of the dual conformal field theory. The relaxation time for the decay of the black hole perturbation is determined by the imaginary part of the lowest quasinormal mode. Our analysis thus establishes a direct relation between this relaxation time and the time scale for return to equilibrium of the dual conformal field theory. This result also provides a new quantitative test of the AdS/CFT correspondence.

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