Finite-Size Effects on Intensity Correlations in Random Media

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The correlations between waves transmitted through random media are analyzed by use of a random-matrix approach and numerical simulations of rough waveguides. Although the intensity and conductance fluctuations are practically independent of the sample length, the correlations present a strong dependence on the length of the disordered region. In waveguide geometries the long-range correlations $C^{(2)}$ and $C^{(3)}$, usually associated to intensity and conductance fluctuations, respectively, become negative as the length of the system decreases. Our results provide a new interpretation of recent optical experiments on disordered slab geometries.

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The propagation of waves through random media produces complex, irregular intensity distributions (*speckle* patterns) [1] which are not nearly as random as intuition suggests. The statistical behavior of such patterns is dominated by the underlying correlations between transport coefficients [2]. Different phenomena such as the enhanced coherent backscattering [3] and intensity correlations in transmitted and reflected electromagnetic and other classical waves [4–13] are closely connected to weak localization and universal conductance fluctuations (UCF) [14–20] in electron transport.

Previous works have been mostly focused in the diffusive regime, where the length of the sample L is much larger than the mean free path ℓ, but still smaller than the localization length ξ. Various types of correlations (usually referred to as $C^{(1)}$, $C^{(2)}$, and $C^{(3)}$) have been identified [4,5,9]. Recently, the existence of a new type of long-range intensity correlation has been pointed out [13]. It was suggested that in quasi-one-dimensional geometries (e.g., disordered waveguides), these new correlations could dominate even the "standard" long-range correlations for small waveguide lengths.

The purpose of this Letter is to analyze the behavior of the different correlations as a function of the length of the disordered region within the macroscopic random matrix theory (RMT) [16,21,22] of multichannel disordered conductors developed by Dorokhov and Mello, Pereyra, and Kumar (DMPK) [23,24]. We also present the results of numerical simulations of wave transport through rough waveguides [25] which are in full qualitative agreement with the predictions based on the DMPK approach. Although the intensity and conductance fluctuations depend weakly on the sample length, we show that the correlations present an unexpected strong dependence on the length of the disordered region. It is remarkable, as we show, that both $C^{(2)}$ and $C^{(3)}$ may be even negative as the length of the system decreases.

The seemingly unrelated problems of the correlation effects in laser speckle patterns and fluctuation phenomena in mesoscopic conductors are tied together by the Landauer formula [18]: The dimensionless conductance G of a waveguide can be defined, both for classical and quantum waves, as the sum over transmission coefficients T_{ab} connecting all input modes a and output modes b, $G = \sum_{ab} T_{ab}$. For disordered mesoscopic systems, the transmission coefficients vary from sample to sample (that is, for different realizations of the locations of defects), so that the variance of conductance fluctuation $\mathrm{var}\{G\} = \langle G^2\rangle - \langle G\rangle^2$ is given by

$$\mathrm{var}\{G\} \equiv= \sum_{ab} \sum_{a'b'} \langle \delta T_{ab} \delta T_{a'b'}\rangle, \qquad (1)$$

where $\langle\cdots\rangle$ denotes the average over an ensemble of samples with different disordered configurations, and $\delta T_{ab} \equiv T_{ab} - \langle T_{ab}\rangle$. $\langle \delta T_{ab} \delta T_{a'b'}\rangle \equiv C_{aba'b'}$ is the correlation function among individual transmission channels. In quantum electronic transport, multiple scattering leads to anomalously large conductance fluctuations (UCF) of the order of $G_0 = 2e^2/h$ (i.e., $\delta G \approx 1$). The physical origin of the UCF is directly related to the statistical properties of the channel (angular) intensity-intensity correlation function $C_{aba'b'}$. The statistical analysis of $C_{aba'b'}$ can be achieved in a light scattering experiment: a laser beam is incident in a given direction (channel) a and the transmitted light intensity can be measured in any direction b (the speckle pattern is just the complex interference pattern in T_{ab} as a function of the outgoing direction b). $C_{aba'b'}$ can then be constructed from the experimental data by collecting T_{ab} for different samples.

The correlation function of the transmission coefficients for scalar wave propagation through disordered media was first calculated by Feng *et al.* [4] by using a diagrammatic technique. Since then, the correlation function is usually written as

$$C_{aba'b'} = C^{(1)}_{aba'b'} + C^{(2)}_{aba'b'} + C^{(3)}_{aba'b'}. \qquad (2)$$

This is a rather general result that does not depend on the details of the scattering mechanism. As a matter of

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fact, essentially the same results are obtained from a RMT approach [5,11]: For a sample with N channels, a length L, and an elastic mean free path ℓ [26], with the definitions $s \equiv L/\ell$, $g \equiv \langle G \rangle$, in a regime where $1 \ll g \lesssim N$, Mello and co-workers [5] showed that

$$\begin{aligned} C_{aba'b'} = {} & \langle T_{ab} \rangle \langle T_{a'b'} \rangle C_1(s) \delta_{aa'} \delta_{bb'} \\ & + \langle T_{ab} \rangle \langle T_{a'b'} \rangle C_2(s) (\delta_{aa'} + \delta_{bb'}) \\ & + \langle T_{ab} \rangle \langle T_{a'b'} \rangle C_3(s), \end{aligned} \tag{3}$$

which has the same structure as that found by Feng *et al.* The fluctuation of different transport parameters is dominated by the different correlation processes. C_1 is the leading term in the correlation function when $a = a'$ (only one incoming channel) and $b = b'$ (only one outgoing channel). Therefore C_1 is the normalized variance of the intensity of a given speckle, $C_1 = \mathrm{var}\{T_{ab}\}/\langle T_{ab} \rangle^2$ sometimes referred to as (far field) *speckle contrast* [1]. C_2 is the leading term in the correlation function when $a = a'$ (only one incoming channel) and $b \neq b'$ (i.e., the correlation between two well separated speckle spots). C_3 is the leading term in the correlation function when $a \neq a'$ and $b \neq b'$. Deep in the diffusive regime, i.e., $1 \ll g \ll N$ the conductance fluctuations are dominated by C_3. As a matter of fact, C_3 is numerically equal to the normalized variance of the conductance, $\mathrm{var}\{G\} \approx g^2 C_3 \approx 2/15$. In the same regime, the C_2 correlations are responsible for the fluctuations of the total transmitted intensity $T_a \equiv \sum_b T_{ab}$, being $C_{2F} \equiv \mathrm{var}\{T_a\}/\langle T_a \rangle^2 \approx 2/(3g) \approx C_2$. Although physically different, the results discussed above may lead one to consider an implicit equivalence between correlations (C_2, C_3) and fluctuations $(C_{2F}, \mathrm{var}\{G\}/g^2)$. However, as shown below, this equivalence holds only deep in the diffusive regime.

Let us first consider the RMT results for the correlations of transmitted waves out of a chaotic cavity. They would roughly correspond to $s = L/\ell \approx 1$ since a mean free path is the characteristic length for phase randomization. To leading order in $N \gg 1$, $\langle T_{ab} \rangle = 1/(2N)$ and $g = N/2$ [21]. The correlations in a chaotic cavity, which have been discussed only in connection with reflected waves [11], have again the same structure as Eq. (3). Following Ref. [11], we found that $C_1 \approx 1$, $C_2 \approx -1/(4g)$, and $C_3 \approx 2/(16g^2)$. These results are strikingly different to those obtained in the diffusive regime. *C_2 is negative,* i.e., just the opposite to the diffusive regime [27]. The fluctuations of the transmitted intensity $[C_{2F} \approx 1/(4g)]$ are essentially dominated by C_1 and C_2, while C_3 is the only relevant correlation behind the conductance fluctuations, $\mathrm{var}\{G\} \approx g^2 C_3 \approx 2/16$ because for this last quantity C_1 and C_2 exactly cancel each other.

These results suggest a subtle interplay among the different correlations as the length of the sample increases which can be analytically investigated within the DMPK approach. By using the method of moments of Mello and Stone [5,17,21] one can compute the variance of both T_a and g. To leading order in $N \gg 1$, the results are [5,17] $\langle T_{ab} \rangle \approx 1/(N\{1+s\})$, $g \approx N/(1+s)$,

$$C_{2F} \equiv \frac{\mathrm{var}\{T_a\}}{\langle T_a \rangle^2} \approx \frac{s^2}{g(1+s)^3} \left(\frac{2}{3} s + 1 \right), \tag{4}$$

$$\mathrm{var}\{G\} \approx \frac{2}{15} \left(1 - \frac{1+6s}{(1+s)^6} \right). \tag{5}$$

Within the same approach we obtain the s dependence of C_1, C_2, and C_3: $C_1 \approx 1$,

$$C_2 \approx \frac{2}{3g} \frac{1}{(1+s)^3} \left(s^3 - 3s - \frac{3}{2} \right), \tag{6}$$

$$\begin{aligned} C_3 \approx {} & \frac{2}{15g^2} \left(1 - \frac{1+6s}{(1+s)^6} \right) \\ & - \frac{1}{g^2(1+s)^4} \left[\frac{4}{3} s^3 + s^2 - 2s - 1 \right]. \end{aligned} \tag{7}$$

The interplay between fluctuations and correlations as a function of the sample length is summarized in Fig. 1. In Fig. 1a we represent gC_{2F} together with gC_2 versus $s = L/\ell$. For small s values C_2 is negative and, only when the system length is larger than a few transport mean free paths, it approaches to C_{2F}. The dashed line in Fig. 1a corresponds to the well known asymptotic value $gC_{2F} \approx gC_2 \approx 2/3$. In Fig. 1b we show the DMPK results for $g^2 C_3$ and $\mathrm{var}\{G\}$ versus s. The dashed line in Fig. 1b corresponds to the well known asymptotic result $g^2 C_3 \approx \mathrm{var}\{G\} \approx 2/15$. C_3 is positive both for small ($s \lesssim 1$) and large ($s \gg 1$) values of s but, interestingly, there is a region in which it may become negative, i.e., within this region two different speckle patterns are not correlated but *anticorrelated*. It is worth noticing that, while $\mathrm{var}\{G\}$ (and, to some extent also C_{2F}) rapidly becomes constant and equal to its asymptotic value (already for $s \approx 1$), the relative weight of the underlying correlations strongly changes with the sample length. This is in striking contrast with the standard interpretation of UCF in terms of long range C_3 correlations.

As expected, the results for a chaotic cavity (open squares in Fig. 1) are very close to those obtained from the DMPK approach for $s \approx 1$. In this regime, flux conservation is the only origin of correlations [28]. This explains why C_2 is negative: for a given incoming angle a, the intensities of two well separated speckle spots (b and b') are anticorrelated (when one is brighter than the average, the other is likely to be darker than the average). After a transition region ($s \approx 2$), where C_2 and C_3 are almost zero and fluctuations are dominated by C_1, the correlations slowly approach their asymptotic values. Deep in the diffusive regime, most of the intensity has been reflected and correlations become dominated by the transmittance of the waveguide (if, for a given realization one speckle spot is darker than average, the others would also be darker than average) [29].

We have performed numerical calculations of the correlation function for waves transmitted through a

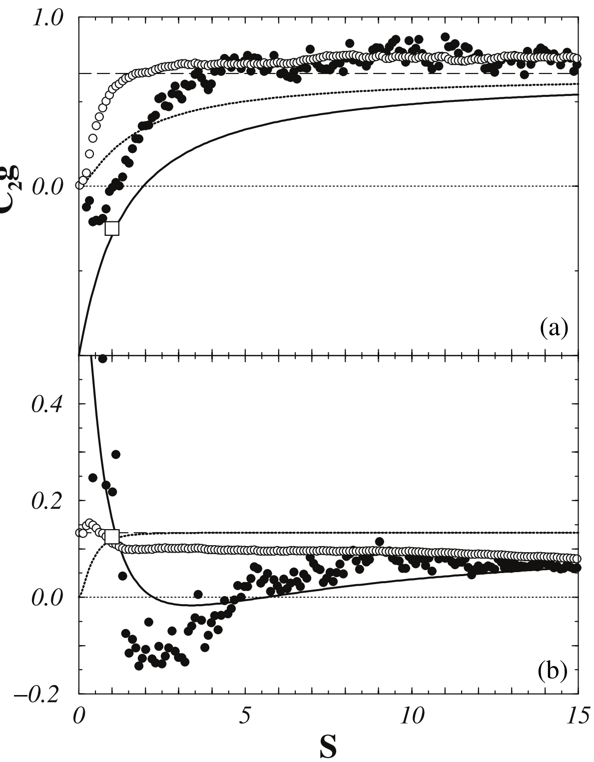

FIG. 1. Dependence of correlations and fluctuations on the waveguide length ($s = L/\ell$) obtained from the DMPK approach (lines) and from numerical simulations of randomly corrugated waveguides (symbols). (a) C_2 correlations (solid lines and full symbols) and C_{2F} intensity fluctuations (dotted line and open symbols). (b) C_3 correlations (solid lines and full symbols) and conductance fluctuations (var$\{G\}$) (dotted line and open symbols). Horizontal lines indicate the asymptotic values: (a) $gC_2 \approx gC_{2F} \approx 2/3$ and (b) $g^2C_3 \approx \mathrm{var}\{G\} \approx 2/15$.

two-dimensional surface disordered waveguide. The corrugated part of the waveguide, of length L and perfectly reflecting walls, is composed of slices of length l. The width of each slice has random values uniformly distributed between $W_0 - \delta$ and $W_0 + \delta$ about a mean value W_0. The model system, as well as the numerical approach, has been described in detail before [25]. In the present work we have taken $W_0/\lambda = 2.6$, allowing $N = 5$ propagating modes and $l/\delta = 3/2$. From our previous work [25], these parameters lead to $\ell \approx 6.3W_0$ and $\xi \approx 34.4W_0 \approx N\ell$. Although the number of modes is small, we found clear support for the behavior predicted by Eqs. (6) and (7). Typical correlations C_2 and C_3 obtained from C_{3234} ($a = a' = 3, b \neq b'$) and C_{3224} ($a \neq a', b \neq b'$) [see Eq. (3)] are shown in Fig. 1 [30] together with C_{2F} and var$\{G\}$. Each symbol corresponds to an average over 1000 independent realizations of the disordered waveguide. As can be seen, the qualitative behavior of the correlations is in full agreement with the predictions based on our RMT approach.

Finally, we discuss the relevance of our predictions in actual experiments. Correlations and fluctuations have been the subject of several experimental studies [7,8,10] (mainly focused on C_{2F}). Most of these studies were carried out deep in the diffusive regime ($s \gg 1$) and good agreement was found between experiment and theory. The strong deviations from the standard results predicted by Eqs. (4)–(7) could be directly observed, for example, in microwave experiments in waveguide geometries with $s \to 1$, i.e., close to the start of the diffusive zone [31]. The situation would be slightly different for typical light experiments. In optical experiments, usually a finite beam of width W is incident onto a slab of thickness L containing a random medium [32]. For highly focused beams ($\ell \approx W \ll L$) recent experiments have found that C_{2F} is significantly smaller than expected [10]. The authors of Ref. [10] proposed a qualitative microscopic explanation based on the idea that the coupling of light into the random media reduces the magnitude of the fluctuations. Here we present an alternative (or complementary) explanation within our macroscopic approach based on the s dependence given by Eq. (4).

For $\ell \ll L \ll W$, the value of C_{2F} for a slab geometry is the same as for a waveguide with $N \approx 2(kW/4)^2$ (the factor of 2 is related to the two polarization states of electromagnetic waves [8]) and $k = 2\pi/\lambda$, i.e., $C_{2F} \approx 4/(kW)^2 L/\ell^*$ [6,20,26]. Reducing W leads to an increase in C_{2F} since N is reduced. In contrast with the waveguide geometry, when $\ell \ll W \ll L$ this does, however, not translate to a corresponding quadratical increase of C_{2F}. It has been shown [6] that for $\ell^* \ll W \ll L$ the fluctuation amplitude increases linearly with the inverse size of the incident beam spot, independently of the sample thickness L [6,20]:

$$\frac{1}{C_{2F}} = \frac{2}{3} k^2 \ell^* W; \qquad \ell^* \ll W \ll L. \tag{8}$$

This behavior can be explained by a simple scaling argument: In a semi-infinite medium an incident beam of width W at the sample surface ($z = 0$) spreads out diffusively inside the sample, $W(z) - W \propto z$, thereby reducing contributions from larger depths z. Only a small region of thickness $L_{\mathrm{eff}} \approx W$ (independent of the actual thickness L) really contributes to C_{2F} [10,19,33]. As a matter of fact, the fluctuations for the slab geometry [Eq. (8)] are the same as for a waveguide, but with an effective thickness $L_{\mathrm{eff}} = (3/8)W$, at least in the diffusive regime ($\ell \ll W$). We make the *ad hoc* assumption that this scaling argument still holds for small beam spots $W \approx \ell$; hence we can rewrite Eq. (4) with $s = 3L_{\mathrm{eff}}/(4\ell^*) = (9/32)W/\ell^*$ as

$$\frac{1}{C_{2F}} \cong 2\left(\frac{8}{9}\right)^2 \frac{[1 + (9/32)W/\ell^*]^2}{1 + (3/16)W/\ell^*} (k\ell^*)^2; \qquad W \ll L. \tag{9}$$

This result yields almost quantitative agreement with the experimental observations [10,20] without any free parameter. For $W \to 0$ the magnitude of the fluctuations is now finite in strong contrast to the classical result, Eq. (8). The inverse amplitude of the fluctuations $1/C_{2F}$

increases linearly with W, with a slope equivalent to the prediction for the diffusive regime, Eq. (8). Furthermore, the limiting value is found to be proportional to ℓ^{*2}, again in perfect agreement with the experiment of Ref. [10]. The absolute values found also match the experimental results. For a quantitative comparison, however, it would be necessary to address the full problem of a finite Gaussian beam incident on a semi-infinite slab.

In conclusion, we have shown that a macroscopic approach is able to capture the physical features of correlations and fluctuations in the transition from the diffusive regime to the regime of a finite but still random system. Our results therefore provide important new insight on how correlations and fluctuations build up in random media.

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[27] Negative C_2 correlations also appear among the reflected waves both within RMT [11] and diagrammatic [12] approaches.
[28] The absence of time-reversal symmetry does not modify the qualitative behavior of fluctuations and correlations of the transmitted speckle pattern.
[29] In agreement with this simple interpretation, the correlations between reflected waves exactly show the oposite behavior: while in the diffusive, or in the localization regimes, C_2 is negative [11,12], C_2 becomes positive for $s < 1$ [A. García-Martín and J. J. Sáenz (unpublished)].
[30] An important hypothesis of the DMPK approach is that the phase factors of the transfer matrix have uniform statistical distribution [5]. This hypothesis "seems reasonable if the system is long compared with the mean free path and the width [5]" (i.e., $s \gg 1$). One of the consequences is that the results are independent on the particular modes involved. Although our numerical correlations are mode dependent [25], the qualitative behavior is in agreement with the DMPK approach, even for $s < 1$.
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[33] Direct evidence for this scale dependence is given in [10,20]. By increasing the thickness L of a slab (with $W =$ const), it was found that the magnitude of the fluctuations saturates rapidly for $L > W$.