Semileptonic $b \to u$ Decays: Lepton Invariant Mass Spectrum

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We compute $\mathcal{O}(\alpha_s^2)$ QCD corrections to the lepton invariant mass spectrum in the decay $b \to ul\nu_l$, relevant for the determination of the Cabibbo-Kobayashi-Maskawa matrix element $|V_{ub}|$. Our method can also be used to evaluate moments of the lepton energy distribution with an $\mathcal{O}(\alpha_s^2)$ accuracy. The Abelian part of our result gives the neutrino invariant mass spectrum in the muon decay and, upon integration, the $\mathcal{O}(\alpha^2)$ correction to the muon lifetime.

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Determination of the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements from precision studies in B physics is one of the main goals of experiments BaBar and Belle, under way at SLAC and KEK. These studies are expected to provide important insights into flavor physics, in particular to shed light on the origin of the CP violation and, possibly, discover "new physics."

The two CKM parameters directly accessible at B factories, $|V_{cb}|$ and $|V_{ub}|$, strongly differ in magnitude, with the former being about 10 times larger than the latter. An accurate determination of $|V_{cb}|$ is much easier since the relevant decay rates are relatively large and the backgrounds are small. For $|V_{ub}|$, the theoretically favorable methods are not feasible experimentally, whereas interpretation of clean experimental signatures suffers from large theoretical uncertainties.

Extraction of $|V_{ub}|$ from inclusive semileptonic decays of B mesons requires a suppression of the much larger contribution of $b \to c$ transitions. In order to do so one has to impose cuts on various observables, and several options have been discussed in the literature. For example, one can select events with large energy of the charged lepton, which can be produced only in $b \to u$ decays, or require that the hadron invariant mass be smaller than the lightest charmed meson D.

Unfortunately, such cuts are so severe that the rate of the remaining events cannot be predicted using the heavy quark expansion. For example, imposing the cut on the electron energy induces a sensitivity of the decay rate to the B-meson light-cone wave function which is not very well known. It can in principle be extracted from measurements of the photon energy spectrum in $b \to s\gamma$. However, the relevant theoretical analysis has been performed only in the limit of an infinite b quark mass and the potentially sizable $\Lambda_{\rm QCD}/m_b$ corrections are not under control. It is desirable, therefore, to have an alternative combination of cuts which can remove the charm background, keep a significant fraction of $b \to u$ events, and preserve the applicability of the standard heavy quark expansion.

Recently, a method fulfilling these requirements has been proposed by Bauer, Ligeti, and Luke [1,2]. Their idea consists in extracting $|V_{ub}|$ from inclusive semileptonic decays $b \to ul\nu_l$ by applying a cut on the invariant mass of the leptons q^2. To eliminate the charm background, one requires $q^2 > q_0^2 = (m_B - m_D)^2 \approx 11.6\ \text{GeV}^2$. It turns out that this cut is mild enough to keep a significant fraction of $b \to u$ transitions. In addition, q_0^2 is sufficiently smaller than m_b^2 for the process to be considered inclusive. Therefore, the heavy quark expansion in $1/m_b$ can be applied with confidence.

Of course, there are several sources of theoretical uncertainties associated with this method, including in roughly equal measure the value of the b quark mass, the nonperturbative power corrections (of third order in the ratio of $\Lambda_{\rm QCD}$ and the characteristic momentum flow), and the two-loop perturbative QCD corrections [3]. The calculation of this last effect is the main purpose of this Letter.

The difficulty connected with such corrections is that they involve the q^2 distribution, rather than the total decay rate. While two-loop corrections to charged particle decays are in general challenging (the first calculations for specific kinematic configurations or the total decay rates have been completed only recently; see, e.g., [4–7]), two-loop corrections to the decay distributions have not been evaluated so far.

In the present calculation we take advantage of the fact that, for the experimentally interesting case, the invariant mass of the leptons is large. We introduce an expansion parameter $\delta = (m_b^2 - q^2)/m_b^2$. In $b \to u$ studies using cuts proposed in [1,2] the maximal value of δ is about 0.5 for $q^2 = q_0^2$. Obviously, increasing q^2 results in a rapid decrease of δ, so that δ can be considered as a small parameter in the region of interest $q^2 > q_0^2$. Therefore, by constructing an algorithm for expanding the relevant Feynman diagrams around $\delta = 0$ and computing several terms of such an expansion, we can derive the $\mathcal{O}(\alpha_s^2)$ correction to the dilepton invariant mass spectrum valid in the region of experimental interest.

Examples of diagrams we have to consider in studying the semileptonic $b \to u$ decay at $\mathcal{O}(\alpha_s^2)$ are shown in Fig. 1. The optical theorem connects the imaginary part of such diagrams with contributions to the decay rate. We first integrate over the lepton and neutrino phase space, thereby reducing the problem to the decay $b \to W^*(q^2)u$, where W^* is a virtual W boson with an invariant mass q^2. In the limit $\delta \to 0$, q^2 approaches m_b^2. Therefore, due to phase space constraints, W^* becomes static. The expansion in δ is constructed by applying the heavy quark/boson expansion to the Feynman diagrams. The only unusual feature in our case is that the initial b quark is on the mass shell. In the heavy quark effective theory (HQET) limit, this leads to propagators of the type $1/(2pk)$, whereas the W^* boson is off-shell so that its propagator has the form $1/(2pk + \delta)$.

Since we are interested in the $\mathcal{O}(\alpha_s^2)$ corrections to the decay distributions, we have to consider the three-loop diagrams of the self-energy type, like those shown in Fig. 1, and extract their imaginary parts. Initially, there are two scales in the problem: using m_b as a unit of energy, these scales can be expressed as $\mathcal{O}(1)$ and $\mathcal{O}(\delta)$. We employ asymptotic expansions to identify contributions arising from these widely separated scales. The region with all loop momenta of $\mathcal{O}(1)$ does not contribute to the imaginary part since it is analytic (polynomial) in δ. When some loop momenta are $\mathcal{O}(\delta)$ and others are $\mathcal{O}(1)$, a three-loop diagram factorizes into a product of one- and/or two-loop diagrams and is easy to evaluate.

The nontrivial part of the calculation is the HQET limit where all loop momenta are of $\mathcal{O}(\delta)$. These diagrams are similar to the three-loop HQET diagrams [8–10] but not identical with them, since some of the lines in the present case are on-shell. We have constructed an algorithm based on recurrence relations and integration-by-parts identities [11] with which one can reduce any relevant three-loop diagram to a linear combination of a few master integrals. Four of these master integrals are new. We compute them in the Euclidean ($p^2 = -1$), $D = 4 - 2\epsilon$ dimensional space.

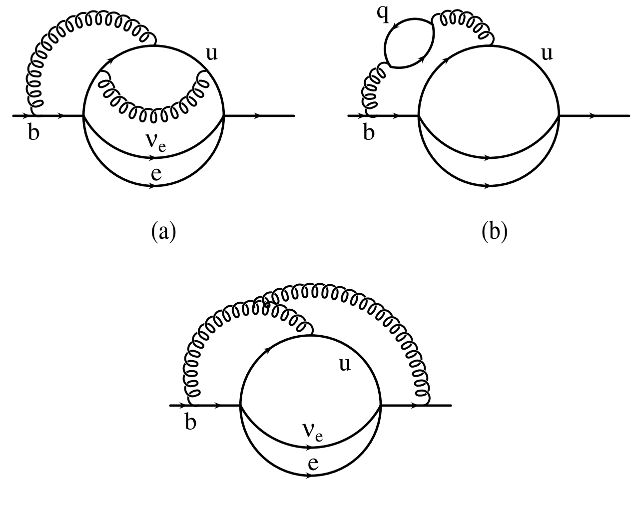

FIG. 1. Examples of diagrams whose cuts contribute to the semileptonic decay $b \to u l \nu_l$: (a) Abelian; (b) light or heavy quarks; (c) non-Abelian.

Propagators occurring in the master integrals are denoted by $D_1 = k_1^2$, $D_2 = k_2^2$, $D_3 = k_3^2$, $D_4 = (k_1 - k_2)^2$, $D_5 = (k_2 - k_3)^2$, $D_6 = 2pk_1$, $D_7 = 2pk_2$, $D_8 = 2pk_3$, $D_9 = 2pk_1 + 2pk_2$, $D_{10} = 2pk_1 + 2pk_3$, $D_{11} = 2pk_1 + 2pk_2 + 2pk_3$. The four new results are

$$I_1 = \int \frac{d^Dk_1\, d^Dk_2}{D_1^\epsilon D_2 D_4 D_6 (D_7 + 1)} = C_\epsilon^2\Big[-\frac{1}{24\epsilon^2} - \frac{5}{24\epsilon} - \frac{13}{24} - \frac{17\pi^2}{48} + \epsilon\Big(\frac{23}{24} - \frac{85\pi^2}{48} - \frac{11}{6}\zeta_3\Big) + \epsilon^2\Big(\frac{623}{24} - \frac{55}{6}\zeta_3 - \frac{95}{16}\pi^2 - \frac{16\,523\pi^4}{8640}\Big) + \mathcal{O}(\epsilon^3)\Big],$$

$$I_2 = \int \frac{d^Dk_1\, d^Dk_2\, d^Dk_3}{D_1 D_2 D_3 D_9 D_{10}(D_{11} + 1)} = C_\epsilon^3\Big[\frac{1}{\epsilon}\Big(-\frac{5}{18} + \frac{\pi^2}{36}\Big) - \frac{9}{2} + \frac{\pi^2}{6} + \frac{7}{3}\zeta_3 + \epsilon\Big(-\frac{91}{2} - \frac{\pi^2}{6} + \frac{11\pi^4}{36} + 14\zeta_3\Big) + \mathcal{O}(\epsilon^3)\Big],$$

$$I_3 = \int \frac{d^Dk_1\, d^Dk_2\, d^Dk_3}{D_1 D_3 D_4 D_5 D_6 D_7 (D_8 + 1)} = C_\epsilon^3\Big[-\frac{\pi^2}{36\epsilon^2} + \frac{1}{\epsilon}\Big(\frac{2}{3}\zeta_3 - \frac{2\pi^2}{9}\Big) + \frac{16}{3}\zeta_3 - \frac{13\pi^2}{9} - \frac{47\pi^4}{270} + \mathcal{O}(\epsilon)\Big],$$

$$I_4 = \int \frac{d^Dk_1\, d^Dk_2\, d^Dk_3}{D_1 D_3 D_4 D_5 D_6 (D_7 + 1) D_8} = C_\epsilon^3\Big[-\frac{\zeta_3}{\epsilon} - 8\zeta_3 - \frac{\pi^4}{60} + \mathcal{O}(\epsilon)\Big]. \tag{1}$$

In the above formulas ζ_3 is the Riemann zeta function, $\zeta_3 = \sum_{i=1}^{\infty} 1/i^3$, and $C_\epsilon \equiv \pi^{2-\epsilon}\Gamma(1 + \epsilon)$.

Using recurrence relations to reduce all loop integrals to a combination of master integrals (these algebraic manipulations are done with FORM [12]), we obtain the $\mathcal{O}(\alpha_s^2)$ correction to the dilepton invariant mass spectrum (we use the pole mass m_b and the $\overline{\mathrm{MS}}$ scheme for α_s)

$$\frac{1}{\Gamma_0}\frac{d\Gamma}{d|\delta|} = 6\delta^2 - 4\delta^3 + \frac{\alpha_s(m_b)}{\pi}X_1 + \Big(\frac{\alpha_s}{\pi}\Big)^2 X_2\,, \tag{2}$$

where $\Gamma_0 = G_F^2|V_{ub}|^2 m_b^5/192\pi^3$ and $X_{1,2}$ denote the one-loop [13] and two-loop corrections, respectively,

$$X_1 = C_F\left[\delta^2\left(\frac{27}{2} - 9L - 4\pi^2\right) + \delta^3\left(\frac{2}{3} - 2L + \frac{8}{3}\pi^2\right) + \delta^4\left(L - \frac{13}{3}\right) - \frac{19}{30}\delta^5 - \delta^6\left(\frac{31}{180} + \frac{L}{6}\right) + \mathcal{O}(\delta^7)\right],$$

$$X_2 = C_F(C_F X_{\rm A} + C_A X_{\rm NA} + T_R N_L X_{\rm L} + T_R N_H X_{\rm H}), \quad (3)$$

where $L = \ln\delta$ and $C_F = 4/3$, $C_A = 3$, and $T_R = 1/2$ are the usual SU(3) color factors, and N_L and N_H denote the number of light ($m_q = 0$) and heavy ($m_q = m_b$) quark species. We use the approximation $m_c = m_b$ since for $q^2 > q_0^2$ there is no phase space available for charm quark production. If needed, corrections for $m_c \neq m_b$ in virtual effects can easily be computed.

For the coefficients $X_{\rm A}, X_{\rm NA}, X_{\rm L}, X_{\rm H}$ we find

$$X_{\rm A} = \delta^2\left\{\frac{27}{4}L^2 - \left(\frac{147}{8} - 4\pi^2\right)L + \frac{523}{16} - \frac{39}{2}\zeta_3 + 7\pi^2\ln2 - \frac{71\pi^2}{6} + \frac{16\pi^4}{15} + \delta\left[\frac{15}{2}L^2 - \left(\frac{287}{12} - \frac{10\pi^2}{9}\right)L + \frac{1363}{72} - \frac{2}{3}\pi^2\ln2 - \frac{217}{108}\pi^2 - \frac{32\pi^4}{45}\right]\right\},$$

$$X_{\rm NA} = \delta^2\left\{\frac{33}{4}L^2 - \left(\frac{423}{8} - \frac{47\pi^2}{6}\right)L + \frac{1103}{16} - \frac{129}{4}\zeta_3 - \frac{7\pi^2}{2}\ln2 - \frac{881\pi^2}{72} + \frac{13\pi^4}{30} + \delta\left[\frac{4}{3}L^2 + \left(\frac{155}{36} - 4\pi^2\right)L - \frac{623}{27} + 24\zeta_3 + \frac{\pi^2}{3}\ln2 + \frac{247\pi^2}{24} - \frac{13\pi^4}{45}\right]\right\},$$

$$X_{\rm L} = \delta^2\left\{-3L^2 + \left(\frac{39}{2} - \frac{8\pi^2}{3}\right)L - \frac{117}{4} + 12\zeta_3 + \frac{41\pi^2}{9} + \delta\left[-\frac{2}{3}L^2 - \left(\frac{31}{9} - \frac{16\pi^2}{9}\right)L + \frac{797}{54} - 8\zeta_3 - \frac{106\pi^2}{27}\right]\right\},$$

$$X_{\rm H} = \delta^2\left[\frac{133}{8} - \frac{5}{3}\pi^2 + \delta\left(-\frac{797}{108} + \frac{2}{3}\pi^2\right) + \delta\left(\frac{2473}{2700} - \frac{1}{18}\pi^2 - \frac{1}{5}L\right)\right]. \quad (4)$$

For brevity we have presented the results accurate up to the terms $\mathcal{O}(\delta^3)$. For the numerical analysis below we use terms up to $\mathcal{O}(\delta^8)$.

We tested these results in several ways. We used a general covariant gauge and checked the cancellation of the gauge parameter. The result for $X_{\rm L}$ agrees with the numerical calculation in [14]. A simple interpolating formula which we actually used for the comparison can be found in the appendix of Ref. [2]. The agreement is very good, practically for all values of δ.

Further, we can extrapolate the results of the expansion by taking the limit $\delta \to 1$ in which case our formulas should describe the decay of a massive quark into a massless quark and a massless W boson. In this limit, second order QCD corrections were computed for the top quark decay [15]. We find that for the color structures $X_{\rm NA,L,H}$ the difference between the two results is better than 10%. The agreement is much worse for the Abelian part $X_{\rm A}$, where the difference can be as large as 50%. This demonstrates that the seven terms of the expansion are insufficient for the Abelian part to converge in the limit $\delta \to 1$.

However, because of the SU(3) color factors, the contribution of the Abelian part is suppressed and we can reliably derive the $\mathcal{O}(\alpha_s^2)$ correction to top quark decay from our formulas. Taking $N_L = 5$ and $N_H = 1$, we find $X_2/2 \approx -16.4$, whereas the central values of the coefficients in [15] give -16.7. An even better agreement is obtained for $\delta = \delta_W \equiv 1 - M_W^2/m_t^2 \simeq 0.79$. At this point, corresponding to physical values of the W boson and top quark masses, the width of $t \to bW$ was evaluated in [16]. We have perfect agreement with the central value of the second order correction, $X_2(\delta_W)/2 = -15.6$, given in Eq. (28) of that paper.

As the final check one can integrate Eq. (2) over δ, obtaining the total decay rate $b \to u l \nu_l$, for which the second order QCD corrections are known [7]. Taking $N_L = 4$ and $N_H = 1$ and integrating over δ we obtain $\int_0^1 d\delta\, X_2(\delta) = -21.24$, in excellent agreement with -21.296, given in Eq. (4) of Ref. [7].

Integrating the abelian contribution $X_{\rm A}$ we can compute the two-photon corrections to the muon lifetime [6,17]. We find $\int_0^1 d\delta\, X_{\rm A}(\delta) \simeq 3.1$, where the 13% discrepancy with the exact value in Eq. (9) of [6] is due to poor convergence of our series for large δ. However, if we assume that the convergence is good up to $\delta \approx 0.65$ and extrapolate for larger δ using $X_{\rm A}(1) = 7.0(4)$ [15], we reproduce the muon lifetime correction [6] within 3%.

For $\delta \lesssim 1/2$, relevant for the extraction of $|V_{ub}|$, the series converge very well and accurately approximate all color components of the $\mathcal{O}(\alpha_s^2)$ correction.

The full $\mathcal{O}(\alpha_s^2)$ correction to the quark decay width, $X_2(\delta)$, is plotted in Fig. 2. Even at the end point $\delta = 1$, our estimate for X_2 agrees with our result for the top decay [15] to better than 3%.

To show the impact of the computed corrections on dilepton invariant mass distribution, we separate the Brodsky-Lepage-MacKenzie (BLM) [18] and non-BLM

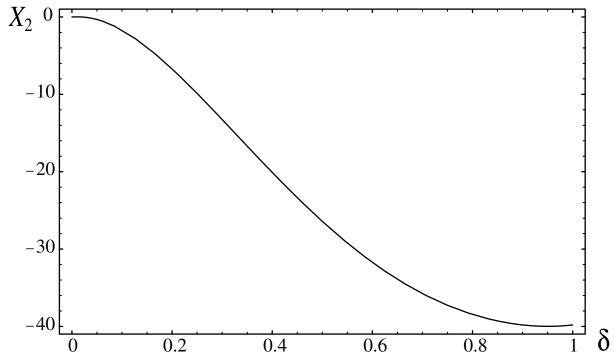

FIG. 2. $\mathcal{O}(\alpha_s^2)$ correction to the decay width $b \to u l \nu_l$, $X_2(\delta)$ [defined in Eq. (3)], as a function of $\delta = 1 - q^2/m_b^2$ (for $N_L = 3$, $N_H = 2$).

corrections since the former have already been studied in the literature. We define the BLM and non-BLM corrections as

$$X_2^{\mathrm{BLM}} = -3 C_F \beta_0 X_L, \qquad X_2^{\mathrm{nonBLM}} = X_2 - X_2^{\mathrm{BLM}}, \tag{5}$$

where $\beta_0 = 11 C_A/12 - T_R N_L/3$ denotes the beta-function coefficient in a theory with three massless quark flavors, appropriate for the range of q^2 used for the V_{ub} extraction.

The value of the BLM corrections is known to be strongly correlated with the scale of the coupling constant used in the one-loop result and also with the quark mass used in the formula for the decay rate. A discussion of these issues can be found in the literature [2] and we will not consider them here. On the contrary, the non-BLM corrections are new. Their dependence on δ is shown in Fig. 3 where the ratio of the non-BLM corrections and the tree level decay rate $6\delta^2 - 4\delta^3$ is plotted. For realistic values of the strong coupling constant, $\alpha_s = 0.2$–0.3, the non-BLM corrections are about 5% in the range of δ relevant for the $|V_{ub}|$ extraction from the dilepton invariant mass spectrum.

The technique described in this Letter might open a way to reliable estimates of the $\mathcal{O}(\alpha_s^2)$ corrections to more complicated observables. For example, a simple modification allows one to calculate the moments of the charged lepton energy spectrum for a fixed value of the dilepton invariant mass.

Recently, combined cuts on both dilepton and hadron invariant masses were advocated for the $|V_{ub}|$ determination [2]. It has been argued that in this approach one can keep the theoretical uncertainties under control while retaining a larger data sample of the $b \to u$ transitions. Since the calculation reported here has been performed without any restriction on the hadronic invariant mass, our results for the QCD corrections are not applicable in this case. However, a sufficiently large number of moments should contain enough information about the spectrum to determine the effect of the cut.

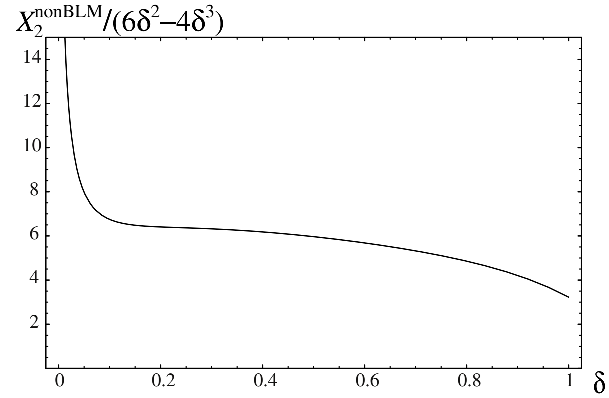

FIG. 3. The non-BLM corrections, $X_2^{\mathrm{nonBLM}}/(6\delta^2 - 4\delta^3)$ (for $N_L = 3$, $N_H = 2$).

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