

## Optimal Eavesdropping in Cryptography with Three-Dimensional Quantum States

D. Bruß<sup>1</sup> and C. Macchiavello<sup>2</sup>

<sup>1</sup>*Institut für Theoretische Physik, Universität Hannover, 30167 Hannover, Germany*

<sup>2</sup>*Dipartimento di Fisica "A. Volta" and INFN-Unità di Pavia, Via Bassi 6, 27100 Pavia, Italy*

(Received 27 June 2001; published 8 March 2002)

We study optimal eavesdropping in quantum cryptography with three-dimensional systems, and show that this scheme is more secure against symmetric attacks than protocols using two-dimensional states. We generalize the according eavesdropping transformation to arbitrary dimensions, and discuss the connection with optimal quantum cloning.

DOI: 10.1103/PhysRevLett.88.127901

PACS numbers: 03.67.Dd, 03.67.Hk

Quantum cryptography, as first suggested by Bennett and Brassard (BB84) [1], is the experimentally most advanced application of quantum information processing. Recently, the use of three-level systems rather than two-level systems for establishing a secure quantum key has been suggested [2]. The authors study the case of 4 mutually unbiased bases, i.e., 12 basis states. They consider an eavesdropper that uses the most simple strategy, namely measuring the state and resending it. For this case they find that a three-dimensional system leads to a higher security than a two-dimensional one.

In order to compare the security of different quantum key distribution protocols, however, one has to study the most general eavesdropping attack. This is the aim of our work. Optimal eavesdropping strategies for the BB84 protocol and the six-state protocol have been studied in [3] and [4,5], respectively.

We concentrate our attention on incoherent attacks; namely, we assume that the eavesdropper interacts with a single three-dimensional quantum system at a time. We study the case where the action of the eavesdropper disturbs all the possible quantum states by the same amount. Denoting with  $\{|0\rangle, |1\rangle, |2\rangle\}$  a basis for the system, the most general unitary eavesdropping strategy for a set of three-dimensional states can be written as

$$\begin{aligned} \mathcal{U}|0\rangle|A\rangle &= \sqrt{1-D}|0\rangle|A_0\rangle + \sqrt{\frac{D}{2}}|1\rangle|A_1\rangle \\ &\quad + \sqrt{\frac{D}{2}}|2\rangle|A_2\rangle, \\ \mathcal{U}|1\rangle|A\rangle &= \sqrt{\frac{D}{2}}|0\rangle|B_0\rangle + \sqrt{1-D}|1\rangle|B_1\rangle \\ &\quad + \sqrt{\frac{D}{2}}|2\rangle|B_2\rangle, \\ \mathcal{U}|2\rangle|A\rangle &= \sqrt{\frac{D}{2}}|0\rangle|C_0\rangle + \sqrt{\frac{D}{2}}|1\rangle|C_1\rangle \\ &\quad + \sqrt{1-D}|2\rangle|C_2\rangle. \end{aligned} \quad (1)$$

Here  $1-D$  is the fidelity of the state that arrives at Bob's site after Eve's interaction. The disturbance is given by  $D$ . We assume the disturbance of the two basis states that are orthogonal to the original to be equal: this symmetry is motivated by the fact that the three basis states should be treated in the same manner. The initial state of Eve's system is called  $|A\rangle$ , and her states after interaction are

labeled  $|A_0\rangle, |B_0\rangle, \dots$  and are normalized. Their dimension is not fixed.

We have to satisfy unitarity of  $\mathcal{U}$ . This leads to the constraints

$$\begin{aligned} \sqrt{\frac{D(1-D)}{2}}(\langle B_0|A_0\rangle + \langle B_1|A_1\rangle) + \frac{D}{2}\langle B_2|A_2\rangle &= 0, \\ \sqrt{\frac{D(1-D)}{2}}(\langle C_2|A_2\rangle + \langle C_0|A_0\rangle) + \frac{D}{2}\langle C_1|A_1\rangle &= 0, \\ \sqrt{\frac{D(1-D)}{2}}(\langle C_1|B_1\rangle + \langle C_2|B_2\rangle) + \frac{D}{2}\langle C_0|B_0\rangle &= 0. \end{aligned} \quad (2)$$

We consider the cryptographic protocol suggested in Ref. [2], where the four mutually unbiased bases are given by  $\{|0\rangle, |1\rangle, |2\rangle\}$ , and

$$\begin{aligned} |\alpha\rangle &= \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle), \\ |\beta\rangle &= \frac{1}{\sqrt{3}}(|0\rangle + \omega|1\rangle + \omega^*|2\rangle), \\ |\gamma\rangle &= \frac{1}{\sqrt{3}}(|0\rangle + \omega^*|1\rangle + \omega|2\rangle); \end{aligned} \quad (3)$$

$$\begin{aligned} |\alpha'\rangle &= \frac{1}{\sqrt{3}}(\omega|0\rangle + |1\rangle + |2\rangle), \\ |\beta'\rangle &= \frac{1}{\sqrt{3}}(|0\rangle + \omega|1\rangle + |2\rangle), \\ |\gamma'\rangle &= \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + \omega|2\rangle); \end{aligned} \quad (4)$$

$$\begin{aligned} |\alpha''\rangle &= \frac{1}{\sqrt{3}}(\omega^*|0\rangle + |1\rangle + |2\rangle), \\ |\beta''\rangle &= \frac{1}{\sqrt{3}}(|0\rangle + \omega^*|1\rangle + |2\rangle), \\ |\gamma''\rangle &= \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + \omega^*|2\rangle); \end{aligned} \quad (5)$$

where  $\omega = e^{2\pi i/3}$ .

We restrict ourselves to the case of symmetric attacks, i.e., Eve is supposed to introduce an equal disturbance to all possible input states written above. [If the noise of the

physical device is known to be symmetric, then Alice and Bob could detect an asymmetric eavesdropper by checking the error rate in a subset of states. Otherwise, the trade-off between Eve's information and the signal key is more complicated to handle.] We can then directly compare the security to the six state scheme for qubits, where only

$$\sqrt{2D(1-D)}(\langle A_1|A_0\rangle + \langle B_1|B_0\rangle + \langle C_2|B_0\rangle + \langle A_1|C_2\rangle) + D(\langle C_1|C_0\rangle + 3\langle B_0|A_1\rangle) = 0, \quad (6)$$

$$\sqrt{2D(1-D)}(\langle B_1|C_0\rangle + \langle A_2|B_1\rangle + \langle A_2|A_0\rangle + \langle C_2|C_0\rangle) + D(\langle B_2|B_0\rangle + 3\langle C_0|A_2\rangle) = 0, \quad (7)$$

$$\sqrt{2D(1-D)}(\langle B_2|B_1\rangle + \langle C_2|C_1\rangle + \langle B_2|A_0\rangle + \langle A_0|C_1\rangle) + D(\langle A_2|A_1\rangle + 3\langle C_1|B_2\rangle) = 0, \quad (8)$$

$$\langle A_1|C_0\rangle + \langle A_2|B_0\rangle + \langle B_0|C_1\rangle + \langle B_2|A_1\rangle + \langle C_1|A_2\rangle + \langle C_0|B_2\rangle = 0. \quad (9)$$

Note that both real and imaginary parts of these expressions have to vanish. Writing the disturbance introduced through the eavesdropping transformation (1) as a function of the scalar products of Eve's states, and taking into account unitarity (2) and the conditions (6)–(9), we find the following simple form:

$$D = 2 \frac{1 - S}{3 - 2S}, \quad (10)$$

where  $S = \text{Re}[\langle A_0|B_1\rangle + \langle B_1|C_2\rangle + \langle C_2|A_0\rangle]/3$ . Notice that in the expression for the disturbance only the scalar products among the eavesdropper's states  $|A_0\rangle$ ,  $|B_1\rangle$ , and  $|C_2\rangle$  appear, while all the others do not contribute.

We will now derive the optimal eavesdropping transformation for a fixed value  $D$  of the disturbance, namely we maximize the mutual information  $I_{AE}$  between Alice and Eve. (This is a standard figure of merit for the description of the efficiency of an eavesdropping attack [3].) As mentioned above, the disturbance introduced by Eve is independent of the scalar products of her states, apart from the ones involving  $|A_0\rangle$ ,  $|B_1\rangle$ , and  $|C_2\rangle$ . Therefore, for any value of  $D$ , Eve is free to choose those states on which  $D$  does not depend in such a way that she retrieves the maximal information. The optimal choice is to take all of these states orthogonal to each other, because in this case Eve can infer the original state sent by Alice in an unambiguous way from her measured state.

We will now consider only the scalar products that appear in  $S$  and choose them such that the mutual information is maximized for fixed  $S$ , i.e., for a given disturbance  $D$ . We introduce the general parametrization for the normalized auxiliary states,

$$\begin{aligned} |A_0\rangle &= x_A|\bar{0}\rangle + y_A|\bar{1}\rangle + z_A|\bar{2}\rangle, \\ |B_1\rangle &= x_B|\bar{0}\rangle + y_B|\bar{1}\rangle + z_B|\bar{2}\rangle, \\ |C_2\rangle &= x_C|\bar{0}\rangle + y_C|\bar{1}\rangle + z_C|\bar{2}\rangle, \end{aligned} \quad (11)$$

where  $\{|\bar{0}\rangle, |\bar{1}\rangle, |\bar{2}\rangle\}$  is an orthonormal basis which is orthogonal to all the other auxiliary states. In order to treat the basis states  $|0\rangle$ ,  $|1\rangle$ ,  $|2\rangle$  in the same way, we require that the overlaps of these three states be equal. We choose

symmetric attacks have been studied. By imposing that the disturbance  $D = 1 - \text{Tr}(|\psi_i\rangle\langle\psi_i|\varrho_B^{\text{out}})$ , where  $\varrho_B^{\text{out}}$  is the reduced density operator of the state sent on to Bob, takes the same value for all 12 possible input states  $|\psi_i\rangle$ , we derive the following relations that involve the scalar products of Eve's output states:

$x_A = y_B = z_C = x$ , while all other coefficients are equal. Without loss of generality we can take the coefficients to be real.

With this strategy we find the optimal mutual information between Alice and Eve to be

$$\begin{aligned} I_{AE} = 1 + (1 - D) &\left[ f(D) \log_3 f(D) \right. \\ &\left. + [1 - f(D)] \log_3 \frac{1 - f(D)}{2} \right], \end{aligned} \quad (12)$$

where  $f(D)$  is given by

$$f(D) = \frac{3 - 2D + 2\sqrt{2}\sqrt{D(3 - 4D)}}{9(1 - D)}. \quad (13)$$

The relation between  $x$  and  $D$  is  $x^2 = f(D)$ . Inserting this into Eqs. (11) leads, together with the ansatz (1) and a straightforward choice of the ancilla states, to the explicit form of the optimal transformation. Eve needs to employ two three-level systems for the optimal attack.

The information for Bob decreases with increasing disturbance:

$$I_{AB} = 1 + (1 - D) \log_3(1 - D) + D \log_3 \frac{D}{2}. \quad (14)$$

Note that we renormalized the functions given in (12) and (14), as in [2], in order to be able to directly relate the values to the two-dimensional case.

We will now compare the security of the three-dimensional scenario as described above with the most secure two-dimensional scheme, that employs six states (i.e., three mutually unbiased bases) [4,5]. The according information curves of both protocols are shown in Fig. 1.

We find that the three-dimensional protocol is more secure in two respects: first, the information curves for Bob and Eve intersect at a higher disturbance  $D_c$  than for the two-dimensional case, namely  $D_{c,3} = 0.227$ , while  $D_{c,2} = 0.156$ . In other words, Eve has to introduce *more* noise in order to gain the same information as Bob. In general, for disturbances  $D < D_c$ , a key distribution protocol

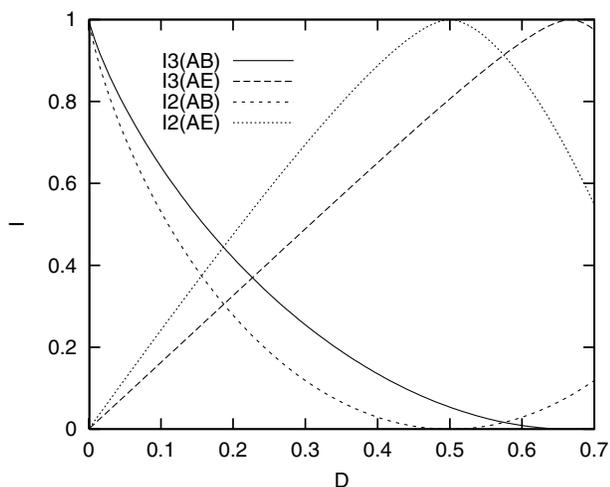


FIG. 1. Mutual information for Alice/Bob and Alice/Eve as a function of the disturbance, for two-dimensional and three-dimensional quantum states.

can be considered secure, because  $I_{AB} > I_{AE}$  [3]. Therefore, the three-dimensional protocol is secure up to higher disturbances. Second, for a fixed disturbance  $D < D_c$ , Bob gets more and Eve less information than in the two-dimensional case. The price that has to be paid for higher security is a lower efficiency: the basis for Bob matches the one of Alice in fewer cases than for two dimensions, as the number of bases is increased.

Notice that our derivation of the optimal eavesdropping transformation relies on equations (6)–(9) which guarantee that all the possible input states are disturbed in the same way. If we reduce the number of bases, not all of these conditions will be necessary, thus leading to a less simple structure of  $D$  than the one given in (10). This would allow a different general form of the optimal eavesdropping transformation, and a higher curve for  $I_{AE}$ . The analogous behavior was shown for the two-dimensional case in [4,5], where the six-state protocol and the BB84 scheme were compared.

Generalizing the ansatz given in (1) and the structure of the ancilla states as in (11) to higher dimensions, we find a lower bound on the eavesdropper's information for quantum cryptography with  $d$ -dimensional systems. The general ansatz is then

$$\begin{aligned} \mathcal{U}|0\rangle|A\rangle &= \sqrt{1-D}|0\rangle|A_0\rangle + \sqrt{\frac{D}{d-1}}|1\rangle|A_1\rangle + \dots, \\ \mathcal{U}|1\rangle|A\rangle &= \sqrt{\frac{D}{d-1}}|0\rangle|B_0\rangle + \sqrt{1-D}|1\rangle|B_1\rangle + \dots, \\ &\vdots \\ \mathcal{U}|d-1\rangle|A\rangle &= \sqrt{\frac{D}{d-1}}|0\rangle|Z_0\rangle + \sqrt{\frac{D}{d-1}}|1\rangle|Z_1\rangle + \dots \end{aligned} \quad (15)$$

(The alphabet denoting Eve's states is supposed to contain  $d$  letters.) The according generalized formula for the disturbance as a function of the scalar products is

$$D = \frac{(d-1)(1-S)}{d-S(d-1)}, \quad (16)$$

where  $S$  is now the real part of the average of all possible scalar products between  $|A_0\rangle, |B_1\rangle, \dots$ . The function  $f$  is then given by

$$f_d(D) = \frac{d-2D + \sqrt{(d-2D)^2 - d^2(1-2D)^2}}{d^2(1-D)}. \quad (17)$$

In Fig. 2 we plot Eve's corresponding information

$$I_{AE,d} = 1 + (1-D) \left[ f_d(D) \log_d f_d(D) + [1-f_d(D)] \times \log_d \frac{1-f_d(D)}{d-1} \right], \quad (18)$$

as a function of the dimension  $d$  for a fixed value of the disturbance  $D$ . We conjecture that this mutual information is optimal when employing the maximal number of mutually unbiased bases for a given dimension [6].

Finally, we discuss the connection between optimal eavesdropping strategies and optimal cloning transformations. The information that Eve can gain is restricted by the laws of quantum mechanics, namely the no-cloning theorem [7]. Let us point out, however, that there is, in general, no direct connection between limits on the cloning fidelity for a given  $d$ -dimensional state, and the intersection of the information curves of Bob and Eve. The reason is that approximate cloning transformations [8] are only a subset of our family of transformations  $\mathcal{U}$  given in Eq. (1), because an additional symmetry between the first of Eve's states and Bob's state is required for cloning. Indeed, if Eve would read only the first of her two states, the disturbance for the intersection between the two resulting information curves would correspond to the fidelity of the optimal cloner. Reading both states increases her information. Therefore, the knowledge of cloning transformations for  $d$ -dimensional systems [9] allows us only to find a lower bound on Eve's information at a given disturbance.

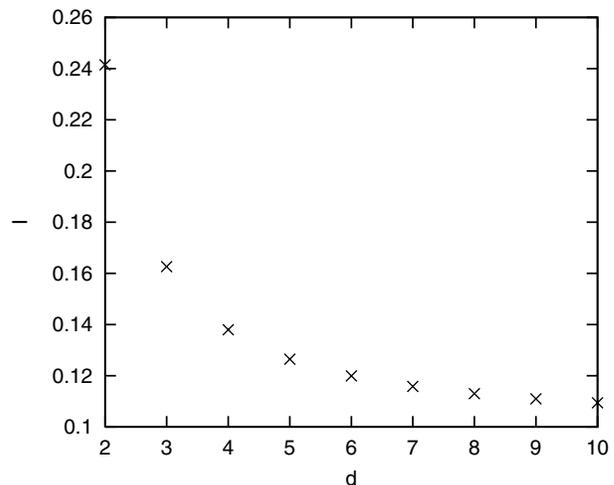


FIG. 2. Mutual information between Eve and Alice as a function of the dimension, for  $D = 0.1$ .

In summary, we have found a remarkable feature of higher-dimensional quantum systems: we have proven analytically for dimension  $d = 3$  that the most general symmetric attack of an eavesdropper gives her less information than in the case of qubits. Therefore a three-dimensional scheme offers higher security than two-dimensional systems. We generalized the upper limit for Eve's information  $I_{AE}$  from  $d = 3$  to higher dimensions: this limit decreases with the dimension, and numerically we find that it reaches  $I_{AE} = D$  in the limit  $d \rightarrow \infty$ . As quantum cryptography is the most advanced technology in quantum information, and security issues play a fundamental role in any study of cryptography, it is important to discuss quantitative properties of the security in quantum key distribution: here quantity becomes quality.

While completing this manuscript we learned about related work by M. Bourennane *et al.* [10].

We wish to thank Maciej Lewenstein for discussions and Silvia Coppola for helpful remarks. This work has been supported by DFG (Schwerpunkt "Quanteninformationsverarbeitung"), the ESF-Programme PESC, and the EU IST-Programme EQUIP.

- [1] C. H. Bennett and G. Brassard, in *Proceedings of the IEEE International Conference on Computers, Systems, and Signal Processing, Bangalore, India* (IEEE, New York, 1984), pp. 175–179.
- [2] H. Bechmann-Pasquinucci and A. Peres, quant-ph/0001083.
- [3] C. Fuchs, N. Gisin, R. Griffiths, C.-S. Niu, and A. Peres, Phys. Rev. A **56**, 1163 (1997).
- [4] D. Bruß, Phys. Rev. Lett. **81**, 3018 (1998).
- [5] H. Bechmann-Pasquinucci and N. Gisin, Phys. Rev. A **59**, 4238 (1999).
- [6] S. Bandyopadhyay, P. Boykin, V. Roychowdhury, and F. Vatan, quant-ph/0103162.
- [7] W. K. Wootters and W. H. Zurek, Nature (London) **299**, 802 (1982).
- [8] V. Bužek and M. Hillery, Phys. Rev. A **54**, 1844 (1996); N. Gisin and S. Massar, Phys. Rev. Lett. **79**, 2153 (1997); D. Bruß, D. P. DiVincenzo, A. Ekert, C. A. Fuchs, C. Macchiavello, and J. A. Smolin, Phys. Rev. A **57**, 2368 (1998); R. Werner, Phys. Rev. A **58**, 1827 (1998).
- [9] S. Albeverio and S.-M. Fei, Eur. Phys. J. B **14**, 669 (2000).
- [10] M. Bourennane, A. Karlsson, G. Björk, N. Gisin, and N. J. Cerf, quant-ph/0106049; N. J. Cerf, M. Bourennane, A. Karlsson, and N. Gisin, Phys. Rev. Lett. **88**, 127902 (2002).