M-Theory Conifolds

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Seven manifolds of G_2 holonomy provide a bridge between M-theory and string theory, via Kaluza-Klein reduction to Calabi-Yau six manifolds. We find first-order equations for a new family of G_2 metrics $\mathbb{D}_7$, with $S^3 \times S^3$ principal orbits. These are related at weak string coupling to the resolved conifold, paralleling earlier examples $\mathbb{B}_7$ that are related to the deformed conifold, allowing a deeper study of topology change and mirror symmetry in M-theory. The $\mathbb{D}_7$ metrics' nontrivial parameter characterizes the squashing of an S^3 bolt, which limits to S^2 at weak coupling. In general the $\mathbb{D}_7$ metrics are asymptotically locally conical, with a nowhere-singular circle action.

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Calabi-Yau manifolds, both compact and noncompact, singular and nonsingular, have long been studied because of their significance for string theory, since they provide a way of obtaining $\mathcal{N} = 1$ supersymmetry in four dimensions. The principal noncompact example is the singular conifold, and its smoothed-out versions, namely, the resolved conifold and the deformed conifold [1]. The singular apex of the cone over $T^{1,1} = (S^3 \times S^3)/S^1$ is blown up to a smooth 2-sphere in the former, and to a smooth 3-sphere in the latter. These minimal (calibrated) surfaces are supersymmetric cycles over which D-branes may be wrapped. If one considers a sequence of smooth models in which the cycles shrink to zero, one obtains enhanced gauge symmetry at the conifold point, the resolved and deformed conifolds describing different regimes of dual strongly coupled field theories [2,3]. Studying this process has led to an understanding of topology change in quantum gravity [4,5].

With the advent of M-theory, it has become important to consider the lifts of these six manifolds with holonomy SU(3) to seven dimensions and holonomy G_2 [2,3,6,7], in order to set the four-dimensional $\mathcal{N} = 1$ theories in an M-theory context. The seven-dimensional and six-dimensional manifolds are related by Kaluza-Klein reduction on a circle, whose variable length R is related to the coupling constant g of type IIA string theory by $R \propto g^{2/3}$. Thus we seek asymptotically locally conical (ALC) G_2 manifolds, for which the size R of the circle tends to a constant at infinity. In the case that R is everywhere constant, and the associated Kaluza-Klein vector field vanishes, the six-dimensional manifold is an exact Calabi-Yau space. If R varies, it will be an approximate Calabi-Yau space. This approximation will be good everywhere if the coupling constant g, or, equivalently, the radius R, never vanishes and is slowly varying. One may show on general grounds that it is never larger than its value at infinity.

Since the principal orbits of the smoothed-out conifold are $T^{1,1}$, it follows that the principal orbits of the associated seven-dimensional G_2 metrics will be a U(1) bundle over $T^{1,1}$, which is in fact $S^3 \times S^3$. Very few examples of cohomogeneity one G_2 metrics can arise [8], and in fact the only explicitly known examples have principal orbits that are $\mathbb{CP}^3$, the flag manifold SU(3)/[U(1) $\times$ U(1)], and $S^3 \times S^3$. Asymptotically conical (AC) metrics are known for all three cases [9,10], but only the $S^3 \times S^3$ case has enough freedom to permit ALC metrics of cohomogeneity one to arise.

In previous work [11], we presented complete nonsingular G_2 metrics, which we denoted by $\mathbb{C}_7$, for which the coupling constant varied in such a finite positive interval. The associated Calabi-Yau space is the Ricci-flat Kähler metric on a complex line bundle over $S^2 \times S^2$ [12,13]. Other work has provided G_2 metrics $\mathbb{B}_7$ associated with the deformed conifold Calabi-Yau space [14–16]. However, in this case the radius R vanishes on an S^3 supersymmetric (calibrated) cycle in the interior. The purpose of this present Letter is to extend the picture by providing a new class of complete nonsingular G_2 metrics, which we denote by $\mathbb{D}_7$, whose associated Calabi-Yau manifold is the resolved conifold. In this case, as in the $\mathbb{C}_7$ metrics, the coupling constant never vanishes. The new metrics provide a unifying link between the deformed and resolved conifolds, via strong coupling and M-theory.

The metrics are invariant under the action of SU(2) $\times$ SU(2), with left-invariant 1-forms σ_i and Σ_1. The metric *Ansatz* is

$$\begin{aligned} ds_7^2 = dt^2 &+ a^2[(\Sigma_1 + g\sigma_1)^2 + (\Sigma_2 + g\sigma_2)^2] \\ &+ b^2(\sigma_1^2 + \sigma_2^2) + c^2(\Sigma_3 + g_3\sigma_3)^2 + f^2\sigma_3^2, \end{aligned} \tag{1}$$

where a, b, c, f, g, and g_3 are functions only of the radial variable t. If we write σ_i in terms of Euler angles, with $\sigma_1 + i\sigma_2 = e^{-i\psi}(d\theta + i\sin\theta d\phi)$, $\sigma_3 = d\psi + \cos\theta d\phi$, and similar expressions using tilded Euler angles for Σ_i, then the M-theory circle is generated by $\psi \to \psi + k$, $\tilde{\psi} \to \tilde{\psi} + k$, where k is a constant. This

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U(1) diagonal subgroup of the right translations is generated by the Killing vector $K = \partial/\partial\psi + \partial/\partial\tilde{\psi}$. [The metric *Ansatz* (1) is a specialization of a nine-function *Ansatz* introduced in [16], in which the metric functions for the $i = 1$ and $i = 2$ directions in the two SU(2) groups are set equal. The diagonal U(1) subgroup of the SU(2) right translations becomes an isometry, as is needed for Kaluza-Klein reduction, under this specialization.] The orbits of SU(2) $\times$ SU(2) are generically six dimensional. In our solutions, the orbits collapse in the interior to a 3-sphere, which in general has a squashed rather than a round SU(2)-invariant metric. The degenerate orbit is known as a bolt; it is a minimal surface and a supersymmetric (associative) 3-cycle.

The metric will have G_2 holonomy, and thus will also be Ricci flat, if it admits a closed and coclosed associative 3-form (see, for example, [9,17,18]; for a recent use of this method, see, for example, [15]). In our case we take

$$\Phi_{(3)} = e^0e^3e^6 + e^1e^2e^6 - e^4e^5e^6 + e^0e^1e^4 + e^0e^2e^5 - e^1e^3e^5 + e^2e^3e^4, \tag{2}$$

where the vielbein is given by $e^0 = dt$, $e^1 = a(\Sigma_1 + g\sigma_1)$, $e^2 = a(\Sigma_2 + g\sigma_2)$, $e^3 = c(\Sigma_2 + g_3\sigma_2)$, $e^4 = b\sigma_1$, $e^5 = b\sigma_2$, and $e^6 = f\sigma_3$. The closure and coclosure imply the algebraic constraints

$$g = -\frac{af}{2bc}, \qquad g_3 = -1 + 2g^2, \tag{3}$$

together with the first-order equations

$$\begin{aligned} \dot{a} &= -\frac{c}{2a} + \frac{a^5f^2}{8b^4c^3}, \\ \dot{b} &= -\frac{c}{2b} - \frac{a^2(a^2 - 3c^2)f^2}{8b^3c^3}, \\ \dot{c} &= -1 + \frac{c^2}{2a^2} + \frac{c^2}{2b^2} - \frac{3a^2f^2}{8b^4}, \\ \dot{f} &= -\frac{a^4f^3}{4b^4c^3}. \end{aligned} \tag{4}$$

Using the closure and coclosure conditions has reduced the Einstein equations, which are of second order and extremely complicated, to a manageable first-order set involving just the four functions a, b, c, and f. One can check that the equations are a consistent truncation of the second-order Einstein equations for the more general nine-function *Ansatz* that was given in [16]. It should be emphasized that although the equations here have reduced to a four-function first-order system, the *Ansatz* is inequivalent to the four-function *Ansatz* introduced in [15]. In particular, the metric *Ansatz* in [15] admits a Z_2 symmetry under which the σ_i and Σ_i are interchanged and the associative 3-form changes sign, while our metric *Ansatz* (1) does not have this symmetry. (We understand that Gukov, Saraikin, and Volovitch are also considering *Ansätze* that break the Z_2 symmetry [19].)

We can find a regular series expansion for the situation where both a and c go to zero at short distance. Substituting the Taylor expansions for the four functions a, b, c, and f into (4), we find

$$\begin{aligned} a &= \frac{t}{2} - \frac{(q^2+2)t^3}{288} - \frac{(31q^4 - 29q^2 - 74)t^5}{69\,120} + \dots, \\ b &= 1 - \frac{(q^2-2)t^2}{16} - \frac{(11q^4 - 21q^2 + 13)t^4}{1152} + \dots, \\ c &= -\frac{t}{2} - \frac{(5q^2-8)t^3}{288} - \frac{(157q^4 - 353q^2 + 232)t^5}{34\,560} + \dots, \\ f &= q + \frac{q^3t^2}{16} + \frac{q^3(11q^2 - 14)t^4}{1152} + \dots, \end{aligned} \tag{5}$$

where, without loss of generality, we have set the scale size so that $b = 1$ on the S^3 bolt at $t = 0$. The parameter q is free and characterizes the squashing of the S^3 bolt along its U(1) fibers over the unit S^2. By studying the equations numerically, using the short-distance Taylor expansion to set initial data just outside the bolt, we find that there is a regular AC solution when $q = 1$ and there are regular ALC solutions for any q in the interval $0 < q < 1$. In fact, the AC solution at $q = 1$ is the well-known G_2 metric on the spin bundle of S^3, found in [9,10]. [One can easily derive this analytically from (4), by noting that it corresponds to the consistent truncation $c = -a$, $f = b$.] The ALC solutions with the nontrivial parameter $0 < q < 1$ are new, and we denote them by $\mathbb{D}_7$. They exhibit the unusual phenomenon of admitting a supersymmetric associative 3-manifold (the bolt) that is not Einstein. The metric function f tends to a constant at infinity, while the remaining functions a, b, and c grow linearly with t; in fact a, b, and c satisfy the first-order equations governing the Ricci-flat Kähler resolved conifold asymptotically at large distance. One can see from (1) that the U(1) Killing vector $K = \partial/\partial\psi + \partial/\partial\tilde{\psi}$ has length given by $|K|^2 = f^2 + c^2(1 + g_3)^2$, and so it follows that its length is nowhere infinite or zero. It ranges from a minimum value $|K| = q$ at short distance to the asymptotic value $|K| = f_\infty$ at infinity.

It may well be that the system (4) is completely integrable, although we have not yet succeeded in finding the general solution to the first-order equations. (By contrast, it is expected that the second-order Einstein equations for Ricci flatness are of the type that would give rise to chaotic behavior [20].) In a somewhat analogous situation in eight dimensions, we did find the general solution to the first-order equations for an *Ansatz* for ALC metrics of Spin(7) holonomy [21]. In that case, the first-order equations could be reduced to an autonomous third-order equation, whose general solution could be given in terms of hypergeometric functions. In the present case, we can again reduce the first-order equations to an autonomous third-order equation for $G \equiv g^2$:

$$\begin{aligned}
&[(-6G^2 + 2G)G'^2 - 4(7G^3 - 2G^2)G' + 8G^3 - 32G^4]G''' + \\
&[(3G + 1)(G')^3 + 6(14G^2 - 3G)G'^2 - 4(9G^2 - 31G^3)G' + 8G^3 - 32G^4]G'' + \\
&[6(3G^2 - G)G' - 12G^2 + 32G^3]G''^2 + 2(3G + 1)G'^4 - 20(G - 6G^2)G'^3 - \\
&8(7G^2 - 29G^3)G'^2 - 16(G^3 + 4G^4)G' = 0, \qquad (6)
\end{aligned}$$

with $A \equiv c^2/a^2 = 1 + G'/(2G)$, $c^2/b^2 = (A' + 2A^2 - 2A)/(G + 3GA - A)$, and $abc = e^{\rho}$. The primes denote derivatives with respect to the new radial variable ρ, defined by $dt = -c\,d\rho$. We have found the following new explicit solution:

$$\begin{aligned}
ds^2 = {}& h^{-1/3}dr^2 + \tfrac{1}{6}r^2h^{-1/3}\Big[\Big(\Sigma_1 + \frac{k}{r}\sigma_1\Big)^2 + \Big(\Sigma_2 + \frac{k}{r}\sigma_2\Big)^2\Big] \\
&+ \tfrac{1}{9}r^2h^{-1/3}\Big[\Sigma_3 + \Big(-1 + \frac{2k^2}{r^2}\Big)\sigma_3\Big]^2 + \tfrac{1}{6}r^2h^{2/3}(\sigma_1^2 + \sigma_2^2) + \tfrac{4}{9}k^2h^{2/3}\sigma_3^2, \qquad (7)
\end{aligned}$$

where $h \equiv 1 - 9k^2/(2r^2)$. Unlike the smooth $\mathbb{D}_7$ metrics that we have found numerically, (7) has a curvature singularity at $r^2 = 9k^2/2$.

It is useful to summarize some known results for G_2 metrics with $S^3 \times S^3$ principal orbits—see Table I.

We are including three families of complete nonsingular solutions here, each of which has a nontrivial parameter. At one end of the parameter range the metric is asymptotically conical. For the $\mathbb{B}_7$ and $\mathbb{D}_7$ cases, this AC metric is precisely the one found in [9,10], on the spin bundle of S^3. Since the bundle is trivial, we are denoting this AC metric simply by $\mathbb{R}^4 \times S^3$. In the case of the $\mathbb{C}_7$ metrics [11], the limiting AC member of the family approaches the form of the AC metric of [9,10] at large distance, but is quite different at short distance, since it instead has the topology of an $\mathbb{R}^2$ bundle over $T^{1,1}$. For the $\mathbb{C}_7$ metrics, and our new $\mathbb{D}_7$ metrics, the nontrivial parameter in the metrics characterizes the degree of squashing of the $T^{1,1}$ or S^3 bolt, respectively, as denoted by the subscripts q on $T^{1,1}_q$ and S^3_q in Table I. By contrast, for the $\mathbb{B}_7$ metrics the S^3 bolt is always round (denoted by the subscript "1" on S^3_1), and the nontrivial parameter instead characterizes "velocities" of the metric functions as one moves outwards from the bolt [16]. (An explicit solution for one specific value of the nontrivial parameter was obtained in [15].)

As the nontrivial parameter in the ALC metric is reduced from its AC limiting value, a circle "splits off" and stabilizes its length when one moves out sufficiently far. The geometry is that of a twisted circle bundle over a six-dimensional AC metric. At the lower limit of the parameter range the radius of the circle at infinity becomes vanishingly small. If one performs an appropriate counterbalancing rescaling of the circle coordinate, the Kaluza-Klein vector describing the twist vanishes in the limit and one obtains the Gromov-Hausdorff limit which is just the direct product of S^1 times a Ricci-flat Calabi-Yau six metric. Thus the Gromov-Hausdorff limit may be identified with the weak-coupling limit in this case. These metrics are listed in the second column of Table I. The metric $\mathbb{C} \ltimes (S^2 \times S^2)$ denotes the Ricci-flat Kähler metric on the complex line bundle over $S^2 \times S^2$ that was constructed in [12,13].

The $\mathbb{B}_7$ and $\mathbb{D}_7$ metrics provide a seven-dimensional link between the six-dimensional deformed and resolved conifolds. This can be seen from the fact that both the $\mathbb{B}_7$ and $\mathbb{D}_7$ families of metrics are encompassed by the *Ansatz* (1). They satisfy two different systems of first-order equations that are each consistent truncations of the same system of six second-order Ricci-flat equations. Each of the $\mathbb{B}_7$ and $\mathbb{D}_7$ families has a continuous nontrivial modulus parameter, with each family having the *same* AC metric at one end of the parameter range, while at the other end of the range the $\mathbb{B}_7$ and $\mathbb{D}_7$ metrics approach S^1 times the deformed conifold and the resolved conifold, respectively. This implies that the two weakly coupled IIA string theory backgrounds using the deformed and resolved conifolds are related via strong coupling and eleven dimensions. [Recently, a metric *Ansatz* more general than (1), with six functions, has been considered, and first-order equations

TABLE I. The three families of G_2 solutions.

G_2 metric	Calabi-Yau	Bolt	AC limit	SUSY cycle?	$\mathbb{Z}_2$ sym?
$\mathbb{B}_7$	Deformed conifold	S^3_1	$\mathbb{R}^4 \times S^3$	Yes	Yes
$\mathbb{C}_7$	$\mathbb{C} \ltimes (S^2 \times S^2)$	$T^{1,1}_q$	$\sim\mathbb{R}^4 \times S^3$	No	Yes
$\mathbb{D}_7$	Resolved conifold	S^3_q	$\mathbb{R}^4 \times S^3$	Yes	No

for G_2 holonomy have been derived [22]. These comprise five first-order equations and one algebraic constraint, and they encompass both the equations (3),(4) obtained here and those obtained previously in [14,15]. The system in [22] thus provides a unified description, entirely within the class of G_2 metrics, of the resolved and deformed conifolds as weak-coupling limits.]

An important issue for future work is the phenomenologically central question of chiral fermions localized at isolated singularities [23–25]. Physically, these can arise in M-theory from massless states associated to membranes wrapped around vanishing cycles. Mathematically, they correspond to solutions of the massless Dirac equation in the M-theory background. The process of localization is as yet imperfectly understood. What is needed is explicit metrics permitting explicit calculations. Our metrics are certainly sufficiently simple for this purpose. What requires further investigation is whether one can model the appropriate codimension seven singularities using them.

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[1] P. Candelas and X. C. de la Ossa, Nucl. Phys. **B342**, 246 (1990).
[2] B. S. Acharya, hep-th/0011089.
[3] M. Atiyah, J. Maldacena, and C. Vafa, J. Math. Phys. (N.Y.) **42**, 3209 (2001).
[4] B. R. Greene, D. R. Morrison, and A. Strominger, Nucl. Phys. **B451**, 109 (1995).
[5] A. Strominger, Nucl. Phys. **B451**, 96 (1995).
[6] B. S. Acharya and C. Vafa, hep-th/0103011.
[7] M. Atiyah and E. Witten, hep-th/0107177.
[8] R. Cleyton and A. Swann, math.DG/0111056.
[9] R. L. Bryant and S. Salamon, Duke Math. J. **58**, 829 (1989).
[10] G. W. Gibbons, D. N. Page, and C. N. Pope, Commun. Math. Phys. **127**, 529 (1990).
[11] M. Cvetič, G. W. Gibbons, H. Lü, and C. N. Pope, hep-th/0111096.
[12] L. Berard-Bergery, C.R. Acad. Sci., Paris, Ser. I **302**, 159 (1986).
[13] D. N. Page and C. N. Pope, Classical Quantum Gravity **4**, 213 (1987).
[14] M. Cvetič, G. W. Gibbons, H. Lü, and C. N. Pope, Nucl. Phys. **B620**, 3 (2002).
[15] A. Brandhuber, J. Gomis, S. S. Gubser, and S. Gukov, Nucl. Phys. **B611**, 179 (2001).
[16] M. Cvetič, G. W. Gibbons, H. Lü, and C. N. Pope, hep-th/0108245.
[17] D. D. Joyce, *Compact Manifolds with Special Holonomy,* Oxford Mathematical Monographs (Oxford University Press, New York, 2000).
[18] S. Salamon, *Riemannian Geometry and Holonomy Groups,* Pitman Research Notes in Mathematics Vol. 201 (Longman, New York, 1989).
[19] S. Gukov (private communication).
[20] J. Demaret, Y. De Rop, and M. Henneaux, Phys. Lett. B **211**, 37 (1988).
[21] M. Cvetič, G. W. Gibbons, H. Lü, and C. N. Pope, Nucl. Phys. **B620**, 29 (2002); math.DG/0105119.
[22] M. Cvetič, G. W. Gibbons, H. Lü, and C. N. Pope, hep-th/0112138.
[23] M. Cvetič, G. Shiu, and A. M. Uranga, Nucl. Phys. **B615**, 3 (2000).
[24] E. Witten, hep-th/0108165.
[25] B. Acharya and E. Witten, hep-th/0109152.