Quantum Instability of a Bose-Einstein Condensate with Attractive Interaction

G. P. Berman, A. Smerzi, and A. R. Bishop

Theoretical Division and Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, New Mexico 87545

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We study the quantum and the mean-field Gross-Pitaevskii (GP) dynamics of a Bose-Einstein condensate gas confined in a toroidal trap. According to GP, if the interatomic interaction is attractive, the rotational states of the system can be dynamically stable or unstable depending on the strength of the mean-field energy. The full quantum analysis, however, reveals that the condensate is *always* unstable. Quantum fluctuations are particularly important close to the GP stability borderline, even for systems with a relatively large number of condensate atoms.

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Background.—According to the mean-field Gross-Pitaevskii picture, homogeneous Bose-Einstein condensate (BEC) gases with an attractive interatomic interaction (negative scattering length) are dynamically unstable. However, if the atoms are spatially confined, the discrete energy spectrum allows a stability window depending on the strength of the mean-field energy. BEC's with a negative scattering length have raised a new class of theoretical problems, including the superfluid nature of such systems [1,2], and the possibility of creating fragmented condensates in double-well [3] and in quasi-one-dimensional toroidal traps [4]. The first experiments with an attractive condensate have been done with an harmonically trapped gas of ^{7}Li alkali atoms [5]. A recent experimental advance has been the demonstrations of tunability of the strength and sign of the interatomic scattering length using a Feshbach resonance [6–8]. This has allowed the investigation of the collapse of an initially stable ^{85}Rb BEC after switching the interatomic interaction from repulsive to attractive. The collapse dynamics has been studied in a wide range of densities/scattering length parameters, and it has offered a number of puzzling results which are still challenging current theoretical models. The Gross-Pitaevskii approach, while predicting the collapse of the system, fails to describe the many facets of its surprisingly rich behavior. It has been argued that three-body inelastic collisions and thermal effects, not included in the GP, can play a crucial role. On the other hand, it is clear that quantum effects can also deeply modify the classical GP trajectories. While a full quantum dynamical analysis is not feasible with the current state of art, some simple models have allowed a first glimpse into the quantum (many-body) dynamics realm. In [9] the collapse dynamics has been studied semiclassically in terms of effective inverted harmonic potentials. It has been found that the growth of the unstable modes is initially delayed, and then becomes superexponentially fast with respect to the GP prediction. Such analysis, however, neglects higher order quantum corrections that are important close to the classical critical point. In [10], it has been argued that, at the onset of the instability, the macroscopic quantum tunneling of the unstable collective modes is the dominant decay mechanism at low temperatures, and simple (but discording) estimates of the tunneling rate have been given in a number of works [10,11].

Here we investigate the dynamical stability of a BEC with an attractive interatomic interaction, and confined in a quasi-one-dimensional (1D) toroidal trap having a square cross section area $S = r_1 r_2$, and radius $R \gg r_{1,2}$ (which fix the radial wave function and justify the 1D assumption). We compare the GP and the quantum dynamical growth of the unstable modes, and we identify the limit in which the quantum corrections of the GP can be neglected. We find that the angular momentum (rotational) states of the BEC are always unstable, and that the GP dynamics is asymptotically recovered in the limit $|a|/d \to 0, N \to \infty$ while keeping $\frac{|a|}{d}N = \text{const}$ ($a < 0$ is the interatomic scattering length, $d = \frac{S}{8R}$ is a characteristic geometric length of the system, and N is the number of condensate atoms). However, the quantum corrections to the GP remain crucial close to the mean-field critical point $\frac{|a|}{d}N = 1$, where they appear on time scales that grow as $\ln(N) \sim \ln(\frac{d}{|a|})$. Interestingly, the $\ln(N)$ scaling has been predicted also at the onset of (different) dynamical instabilities [12].

Classical dynamics.—At the mean-field GP level, the collapse of a condensate with an attractive interaction is driven by a modulational (or parametric) instability (MI). MI is a general feature of discrete as well as continuum nonlinear wave equations, and refers to an exponential growth of small fluctuations of a carrier wave, as a result of the interplay between dispersion and nonlinearity.

Here we consider a Bose-Einstein condensate trapped in an effective 1D toroidal geometry, in regimes of small nonlinearity $|a|/d \ll 1$. At sufficiently low temperatures the system is described, at a mean-field level, by the GP equation, which (in dimensionless units) is

$$i \frac{\partial \Psi}{\partial t} = \left[-\frac{\partial^2}{\partial \theta^2} + U|\Psi|^2 \right] \Psi, \tag{1}$$

with $U = \pi \frac{a}{d}$ and $0 < \theta \leq 2\pi$. The time has been rescaled as $\frac{\hbar}{2mR^2} t \to t$, and the normalization of the wave function is $\int |\psi(\theta)|^2 \, d\theta = N$. Expanding in plane waves,

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$\Psi(\theta,t) = \frac{1}{\sqrt{2\pi}}\sum_q c_q(t)e^{iq\theta}$, with q integer, Eq. (1) becomes

$$i\dot{c}_q(t) = q^2 c_q(t) + \frac{U}{2\pi}\sum_{mln} c_n^*(t)c_m(t)c_l(t)\delta_{m+l-n-q}\,, \tag{2}$$

with $c_q(t), c_q^*(t)$ conjugate variables in the Hamiltonian: $H = \sum_q q^2 c_q^*(t)c_q(t) + \frac{U}{4\pi}\sum_{mlnq} c_n^*(t)c_q^*(t)c_m(t)c_l(t) \times \delta_{m+l-n-q}$. Solutions of Eq. (2) are the finite amplitude waves:

$$c_q(t) = c_q \exp\Big[-i\Big(q^2 + \frac{U}{2\pi}|c_q|^2\Big)t\Big]\delta_{q,k}\,, \tag{3}$$

with $k = 0, \pm 1, \pm 2, \ldots$, and initial conditions $c_q(t=0) = c_q\delta_{q,k}$. The states (3) can be unstable against a four-phonon decay process: $2k \rightarrow (k+p) + (k-p)$. With $|c_k|^2 = N \gg |c_{k+p}|^2 \sim |c_{k-p}|^2$, Eq. (2) can be linearized:

$$\begin{aligned} i\dot{c}_{k+p}(t) &= \Big[(k+p)^2 + \frac{U}{\pi}|c_k|^2 - \mu\Big]c_{k+p}(t) + \frac{U}{2\pi}c_k^2(t)c_{k-p}^*(t) \\ i\dot{c}_{k-p}(t) &= \Big[(k-p)^2 + \frac{U}{\pi}|c_k|^2 - \mu\Big]c_{k-p}(t) + \frac{U}{2\pi}c_k^2(t)c_{k+p}^*(t)\,, \end{aligned} \tag{4}$$

with the chemical potential $\mu = k^2 + \frac{U}{2\pi}|c_k|^2$. The eigenfrequencies of (4) are $\omega_\pm = 2kp \pm |p|\sqrt{p^2 - \frac{|U|}{\pi}|c_k|^2}$. Therefore, the system becomes modulationally unstable, with an exponential growth of $k \pm p$ modes, when

$$\frac{|U|}{\pi}|c_k|^2 = \frac{|a|}{d}N > p^2; \qquad p = \pm 1, \pm 2, \ldots\,. \tag{5}$$

In [4] it has been shown that this modulational instability leads to the fragmentation of the condensate over different angular momentum states. In the following we study the onset of this hybridization at a full quantum level. In [9] quantum corrections have been included in Eq. (4) quantizing the $c_{k\pm p}$ modes while leaving the large wave c_k as a classical (c-number) field. This corresponds to neglect higher order quantum correlations between the $k, k \pm p$ fields. Such correlations, however, are crucial when the system is close to the classical instability point and will not be neglected in our analysis.

Quantum dynamics.—The many-body dynamics of the system is governed by the quantum field equation

$$i\frac{\partial\hat{\Psi}}{\partial t} = \Big[-\frac{\partial^2}{\partial\theta^2} + U\hat{\Psi}^\dagger\hat{\Psi}\Big]\hat{\Psi}\,. \tag{6}$$

With $\hat{\Psi}(\theta,t) = \frac{1}{\sqrt{2\pi}}\sum_q \hat{a}_q(t)e^{iq\theta}$, we have

$$i\dot{\hat{a}}_q = q^2\hat{a}_q + \frac{U}{2\pi}\sum_{mln}\hat{a}_n^\dagger\hat{a}_m\hat{a}_l\delta_{m+l-n-q} \tag{7}$$

and $\hat{H} = \sum_q q^2\hat{a}_q^\dagger\hat{a}_q + \frac{U}{4\pi}\sum_{mlnq}\hat{a}_n^\dagger\hat{a}_q^\dagger\hat{a}_m\hat{a}_l\delta_{m+l-n-q}$. We note that the use of the Fermi-pseudopotential in 1D does not introduce the well-known divergency problems found in higher dimensions. The validity of the 1D many-boson Hamiltonian has been discussed in [13,14], and the theory of the zero-range potentials in arbitrary dimensions has been presented in [15].

To study Eq. (7) we use a projection on a coherent state basis $|\alpha_q\rangle$ method described in [16]. The dynamical evolution of $\alpha_q(t) = \langle\alpha_q|\hat{a}_q(t)|\alpha_q\rangle$ is given by an exact (and closed) c-number equation:

$$i\dot{\alpha}_q(t) = \hat{K}\alpha_q(t)\,, \tag{8}$$

with

$$\begin{aligned} \hat{K} = & \sum_q q^2\Big(\alpha_q\frac{\partial}{\partial\alpha_q} - \text{c.c.}\Big) \\ & + \frac{U}{2\pi}\sum_{\{q_i\}}\Big(\alpha_{q_1}\alpha_{q_2}\alpha_{q_3}^*\frac{\partial}{\partial\alpha_{q_4}} - \text{c.c.}\Big)\delta_{q_1+q_2-q_3-q_4} \\ & + \frac{U}{4\pi}\sum_{\{q_i\}}\Big(\alpha_{q_1}\alpha_{q_2}\frac{\partial}{\partial\alpha_{q_3}}\frac{\partial}{\partial\alpha_{q_4}} - \text{c.c.}\Big)\delta_{q_1+q_2-q_3-q_4} \end{aligned} \tag{9}$$

and initial conditions $\alpha_q(t=0) = \alpha_q$. Solutions of Eq. (8) are the finite amplitude waves [c.f. Eq. (3)]:

$$\alpha_q(t) = \alpha_q\exp[-iq^2t + (e^{-i\frac{U}{2\pi}t} - 1)|\alpha_q|^2]\delta_{q,k}\,. \tag{10}$$

These waves exhibit reversible collapses and revivals of the many-body coherence induced by the interaction. We now study the quantum stability of (10) with respect to the four-phonons decay $2k = (k+p) + (k-p)$, and we will compare the results with the GP classical stability condition Eq. (5). At $t = 0$, let the system be excited as $|\alpha_k|^2 = N \gg |\alpha_q|^2, (q \neq k)$. This scenario can be experimentally realized by inverting the sign of the scattering length of a stable, large condensate, as in [7]. The generic mode $\alpha_{k\pm p}(t)$ can be expanded over α_q [17]:

$$\alpha_{k+p}(t) = f_0^{(p)}(\alpha_k,\alpha_k^*,t) + \sum_{q\neq 0}[f_q^{(p)}(\alpha_k,\alpha_k^*,t)\alpha_{k+q} + \tilde{f}_q^{(p)}(\alpha_k,\alpha_k^*,t)\alpha_{k+q}^*] + O(|\alpha_{k+q}|^2)\,, \tag{11}$$

with the following initial conditions: $\tilde{f}(0) = 0$; $f_q^{(p)}(0) = \delta_{p,q}$ and $f_0^{(p)}(0) = 0$ if $p \neq 0$; $f_q^{(p)}(0) = 0$ and $f_0^{(p)}(0) = \alpha_k$ if $p = 0$. Substituting and collecting terms of the same power of α_{k+q} we obtain a close chain of equations for the expansion coefficients

$$i\dot{f}_0^{(p)} = \hat{M} f_0^{(p)}; \quad i\dot{f}_q^{(p)} = \hat{M} f_q^{(p)} + \left[(k+q)^2 + \frac{U}{\pi}|\alpha_k|^2\right] f_q^{(p)} + \frac{U}{\pi}\alpha_k \frac{\partial}{\partial \alpha_k} f_q^{(p)} - \frac{U}{2\pi}\alpha_k^{*2}\tilde{f}_{-q}^{(p)};$$
$$i\dot{\tilde{f}}_{-f}^{(p)} = \hat{M}\tilde{f}_{-q}^{(p)} + \left[-(k-q)^2 - \frac{U}{\pi}|\alpha_k|^2\right]\tilde{f}_{-q}^{(p)} - \frac{U}{\pi}\alpha_k \frac{\partial}{\partial \alpha_k}\tilde{f}_{-q}^{(p)} + \frac{U}{2\pi}\alpha_k^2 f_q^{(p)}, \tag{12}$$

with $\hat{M} = (k^2 + \frac{U}{2\pi}|\alpha_k|^2])\alpha_k \frac{\partial}{\partial \alpha_k} + \frac{U}{4\pi}\alpha_k^2 \frac{\partial^2}{\partial \alpha_k^2}$ − c.c. The solution of the first equation in (12) is given by (10) and describes the dynamics of a large wave in first order perturbation theory. The system of the other two equations can be simplified since, due to the initial conditions and the linearity of Eq. (12), only the two components $f_p^{(p)}(\alpha_k, \alpha_k^*, t)$ and $\tilde{f}_{-p}^{(p)}(\alpha_k, \alpha_k^*, t)$ are different from zero, with $\alpha_{k+p}(\alpha_k, \alpha_k^*, t)(t) = f_p^{(p)}(\alpha_k, \alpha_k^*, t)\alpha_{k+p} + \tilde{f}_{-p}^{(p)}(\alpha_k, \alpha_k^*, t)\alpha_{k-p}^*$. Writing $f_p^{(p)} = f^* e^{i[(k-p)^2 - 2k^2 - \frac{U}{2\pi}]t}$, $\tilde{f}_{-p}^{(p)} = \frac{\alpha_k}{\alpha_k^*} g^* e^{i[(k-p)^2 - 2k^2 - (U/2\pi)]t}$, and replacing in (12), we obtain that f, g obey the system of coupled equations

$$i\frac{\partial f}{\partial \tau} = 2(x-1)f - xg + 2\gamma x \frac{\partial f}{\partial x},$$
$$i\frac{\partial g}{\partial \tau} = -xf, \tag{13}$$

with initial conditions $f(0, x) = 1$, $g(0, x) = 0$, and where $x = |\alpha_k|^2 \frac{|U|}{2\pi}(|\frac{U}{4\pi} - p^2|)^{-1}$, $\tau = \frac{\hbar}{2mR^2}(|\frac{U}{4\pi} - p^2|)t$, $\gamma = \frac{|U|}{2\pi}(|\frac{U}{4\pi} - p^2|)^{-1}$.

The GP limit of Eq. (8) [and, therefore, of (13)], is recovered when $\gamma \sim \frac{|a|}{d} \to 0, |\alpha_k|^2 = N \to \infty$ such that $x \to \frac{|U|}{2\pi}|\alpha_k|^2 \sim \frac{|a|}{d}N \to$ const. In this limit Eqs. (13) can be solved analytically giving $f(x, \tau) = e^{i\tau(1-x)}[\cosh(\sqrt{2x - 1}\tau) + i\frac{1-x}{\sqrt{2x-1}}\sinh(\sqrt{2x - 1}\tau)]$; $g(x, \tau) = -ie^{i\tau(1-x)}\frac{x}{\sqrt{2x-1}}\sinh(\sqrt{2x - 1}\tau)$. The oscillation frequencies of the system becomes imaginary when $x > x_c = \frac{|U|}{2\pi}N = \frac{1}{2}$, which agrees with the instability condition given by Eq. (5).

In Fig. (1) we show the classical (dashed line) and quantum (solid line) evolution of $|\alpha_{k+p}|^2$ as a function of time. The quantum dynamics has been obtained solving Eqs. (13) numerically. In (a) and (b), $x = 0.45$ and $x = 0.49$, respectively, which are in the classical stable region. However, while the classical trajectories remain confined, the quantum evolution of $|\alpha_{k+p}|^2$ first collapses due to a loss of coherence [in a time scale approximately given by Eq. (10)] and then explodes due to the instability. The explosion becomes faster on approaching the mean-field critical instability value $x_c = 0.5$. In Figs. 1(c) and 1(d), $x = 0.51$ and $x = 0.55$, and also the classical trajectories become unstable. Yet, the quantum trajectory grows faster than the classical one, especially when x is sufficiently close to the critical value x_c. A further view is provided in Fig. (2), where we plot the phase portrait Re$[\alpha]$ − Im$[\alpha]$ of the classical (left column) and quantum (right column) trajectories with $x = 0.3, 0.45$, i.e., in the classically stable region. The initial loss of coherence of the system is dramatically demonstrated by the spiraling down of the quantum trajectory to a single point (c), which eventually explodes. Closer to the instability point (d), the time scale associated with the instability growth becomes too large to observe any dephasing effect. In Fig. (3) we plot τ_0 as a function of x for different values of γ; τ_0 is defined as the time when $|\alpha_{k+p}(\tau_0)|/|\alpha_{k+p}(\tau = 0)| = 10$. The classical solution $\gamma = 0$ diverges for $x \to 0.5$, sharply separating the $\tau_0 - x$ plane into stable and unstable regions. The quantum solutions, on the other hand, show that the system is always unstable, and eventually collapses in a finite time, as already discussed. Here, however, it becomes clear how the quantum trajectories reduce to the GP ones when $\gamma \sim \frac{|a|}{d} \to 0$. For realistic experiments (see section below), if $x \ll x_c$ the collapse time can be too large to be observed, and if $x \gg x_c$, the difference between classical and quantum trajectories would be too small. On the other hand, the difference between the GP ($\gamma = 0$) and the quantum trajectories ($\gamma \neq 0$) remains important when x is sufficiently close to the critical value also for a relatively small value of γ. In the inset we show the linear grow of τ_0 as a function of $\ln(N)$ [$\sim \ln(d/|a|)$]. We emphasize that this scaling is predicted for the value $x = 0.49$, i.e., in the classical stable region.

Numerical estimates.—To discuss some useful orders of magnitude, we consider a 1D torus geometry with a

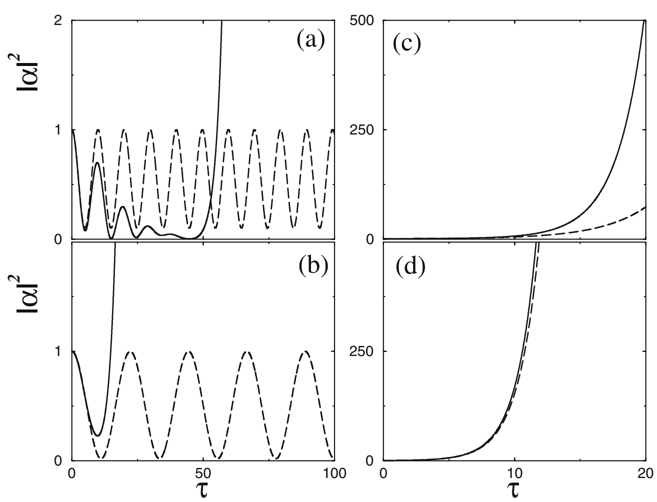

FIG. 1. $|\alpha|^2 = |\alpha_{k+p}(\tau)|^2/|\alpha_{k+p}(\tau = 0)|^2$ as a function of time for $\gamma = 10^{-3}$ and $x = 0.45$, (a), 0.49 (b), 0.51 (c), and 0.55 (d). The dashed line and the solid line are, respectively, the classical ($\gamma = 0$) and quantum ($\gamma \neq 0$) solutions of Eq. (13).

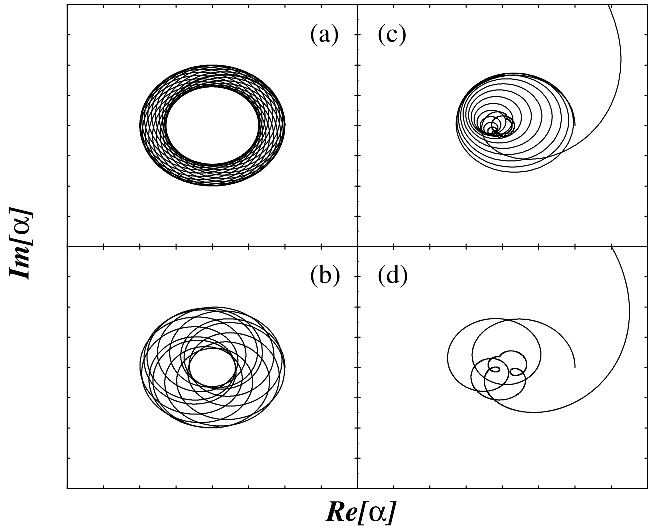

FIG. 2. Phase portrait of the classical (a),(b) and quantum (c),(d) trajectories with $x = 0.3$ (a),(c), and $x = 0.45$ (b),(d), and $\alpha = \alpha_{k+p}(\tau)/\alpha_{k+p}(\tau = 0)$.

$N = 1000$ ^{87}Rd condensate atoms. A progress report on the experimental realization of a toroidal traps is given in [18], while the possibility of realizing a mesoscopic one has been discussed in [19]. Consistent with [19], we choose the cross section area of the torus $S = r_1 r_2 \sim 10^{-8}$ cm^2, and the radius $R \sim 5 \times 10^{-4}$ cm. The nonlinear coupling can be tuned over different orders of magnitude, and, at the critical point, $|a| = 2.5 \times 10^{-6}$ cm and $\gamma \sim 2 \times 10^{-3}$. The growth time of the quantum fluctuations scales as $t = \frac{8mR^2}{\hbar|(U/\pi) - 4p^2|}\tau \sim \frac{67}{p^2}\tau$ ms, and, from Fig. (3), it can be argued that the system dephases in $\sim$800 ms considering the lowest $p = 1$ unstable mode. This time can be significantly reduced with traps of smaller radius R, and/or with lighter alkali atoms. The growth of the unstable modes can be experimentally measured with interferometry techniques in a series of destructive measurements, allowing the condensate to expand and overlap after switching off the toroidal trap [20]. The dynamical growth of the quantum fluctuations is manifested on the gradual loss of the fringes' contrast. We note that our analysis describes the onset of the instability, while for large times [see Eq. (11)], other mechanisms not included here (such as three-body recombinations) will clearly affect the collapse dynamics.

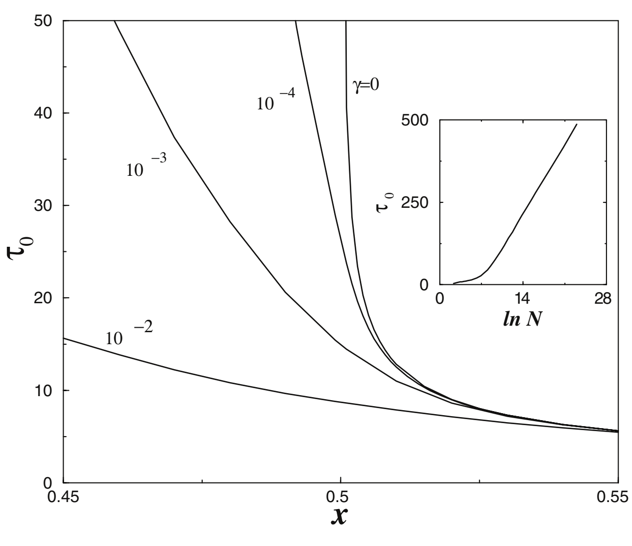

FIG. 3. Dependence of τ_0 as a function of x. τ_0 is defined as the time when $|\alpha(\tau_0)|^2 = |\alpha_{k+p}(\tau_0)|^2/|\alpha_{k+p}(\tau = 0)|^2 = 10$. The values of γ are 10^{-2}, 10^{-3}, 10^{-4}, and 0. Inset: Dependence of τ_0 as a function of $\ln(N)$ with $x = 0.49$.

Conclusions.—Quantum corrections to the Gross-Pitaevskii equation are crucial when the system is close to an instability point. The method (and the results) described here can have far reaching consequences on the study of dynamical quantum chaos in many-body systems, and trapped BEC's provide an ideal experimental situation in this field where still very few results are known [16].

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