Percolation Model for Nodal Domains of Chaotic Wave Functions

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Nodal domains are regions where a function has definite sign. In [1] it is conjectured that the distribution of nodal domains for quantum eigenfunctions of chaotic systems is universal. We propose a percolationlike model for description of these nodal domains which permits us to calculate all interesting quantities analytically, agrees well with numerical simulations, and due to the relation to percolation theory opens the way to deeper understanding of the structure of chaotic wave functions.

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In a recent paper [1], Smilansky *et al.* consider the following problem. Let $\Psi(x, y)$ be a real eigenfunction of a two-dimensional quantum problem. The equation $\Psi(x, y) = 0$ determines a set of nodal lines which separate nodal domains where $\Psi(x, y)$ is of opposite signs. In [1] it is argued that the distribution of the number of these regions for high excited states (i) is universal for integrable as well as for chaotic models but (ii) clearly distinguishes between these two types of models.

For chaotic (billiard) systems it is conjectured in [1] that this distribution coincides with the distribution of nodal domains for Gaussian random functions which are known to give a good universal description of wave functions of chaotic systems with a given energy E [2]:

$$\Psi(x, y) = \sum_{m=-\infty}^{\infty} C_m \Psi_m^{(0)}(x, y) . \tag{1}$$

Here $\Psi_m^{(0)}(x, y) = J_{|m|}(kr)e^{im\phi}$ form the standard basis for billiard problems, $k = \sqrt{E}$ is the momentum, and $C_m = C_{-m}^*$ are independent random variables with Gaussian distribution. Numerical calculations of the nodal domain distribution for these functions and for certain chaotic systems have been performed in [1].

The purpose of this Letter is to demonstrate that nodal domains of random functions (1) (and, consequently, wave functions of generic chaotic systems [1,2]) can be described by a simple percolationlike model where all interesting quantities can be calculated analytically. The model permits us also to apply ideas and methods developed within the percolation theory to the field of quantum chaos and to other problems where nodal domains of random functions are of importance, e.g., to the description of the interface between two-phase systems (see, e.g., [3] and references therein).

To understand how the nodal domains look we give in Fig. 1 their picture for one realization of random function (1). White and gray regions correspond to domains where function $\Psi(x, y)$ has different signs. The figure contains 907 connected nodal domains of different sizes and various irregular shapes.

The mean number of zeros of random functions (1) along a straight line (say the vertical one) can be estimated by noting that, if the size in the y direction is L_y, the approximate quantization condition reads $\bar{k}_y L_y \approx \pi m$ where m is an integer and $\bar{k}_y$ is the mean square momentum along the y axis, $\bar{k}_y^2 = k^2/2$. Therefore

$$\bar{\rho}(y) = \frac{m}{L_y} = \frac{k}{\pi\sqrt{2}} . \tag{2}$$

The rigorous derivation of this relation using the method of [4] will be given elsewhere [5]. The same answer, of course, can be obtained for the mean density of nodal lines along any other straight lines [6] and nodal lines of such random functions can in the *mean* be considered as forming an approximate rectangular lattice whose total number of sites is asymptotically

$$N_{\text{tot}} \approx \frac{k^2 A}{2\pi^2} = \frac{2}{\pi} \bar{N}(E) , \tag{3}$$

where A is the area of the billiard and $\bar{N} = AE/4\pi$ is the mean number of levels with energy less than E.

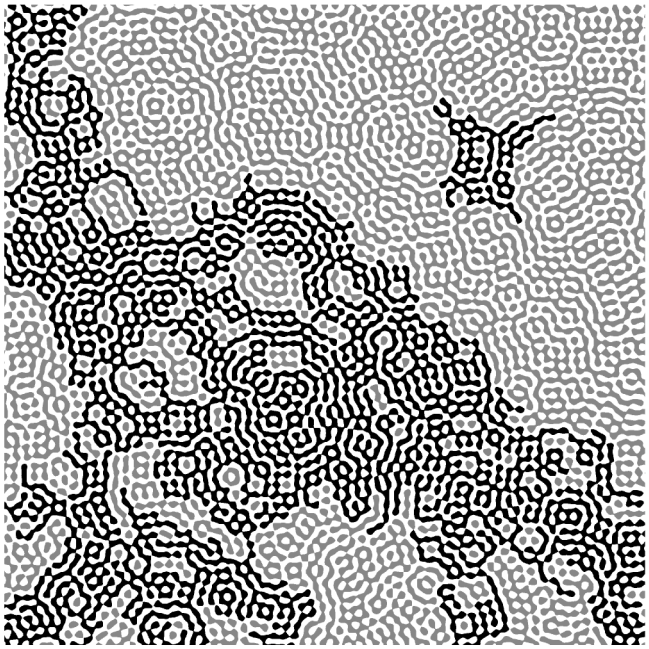

FIG. 1. Nodal domains for a realization of random function (1) with $k = 100$ in a square window of size $L = 4$. The largest of these domains and the largest of the domains which do not touch the boundary are highlighted for clarity.

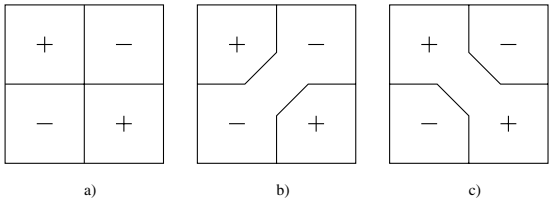

FIG. 2. (a) True nodal crossing. (b) and (c) Avoided nodal crossings.

But this simple picture can be valid only in the mean. When $\Psi(x,y) = \bar{\Psi}(x,y) + \delta\Psi(x,y)$ where function $\bar{\Psi}(x,y)$ has a crossing of nodal lines as in Fig. 2a, the addition of a small correction $\delta\Psi(x,y)$ changes, in general, the true crossing to one of two possible avoided crossings as in Figs. 2b and 2c.

Consequently, one can conjecture that the distribution of nodal domains for random functions is the same as for the following random percolationlike process. Let us consider a rectangular lattice with the total number of sites N_{tot} as in (3). Each line crossing with probability $1/2$ is changed either to the avoided crossing as in Fig. 2b or to the one as in Fig. 2c. These rules give a well-defined random percolationlike process. One realization of such a process is presented in Fig. 3.

The original lattice gives rise to two dual lattices called below positive and negative whose vertices are in the centers of regions where our function is positive or negative (see Fig. 3) and whose size a coincides with de Broglie wavelength [cf. (2)]: $a = 2\pi/k$. Any realization of the above mentioned random process uniquely defines two graphs on these lattices (which we call also positive and negative) with the following properties: (i) their vertices coincide with the vertices of the corresponding lattice; (ii) their edges join together the connected components of this lattice. (A point is also a component of the graph.)

One can choose arbitrarily a graph on one lattice (say negative) and any of such graphs will correspond to an allowed realization and vice versa. Therefore our random process is determined mostly by the bond percolation model on one of the dual lattices (see, e.g., [7]) where with probability $1/2$ one connects 2 nearby sites by a bond.

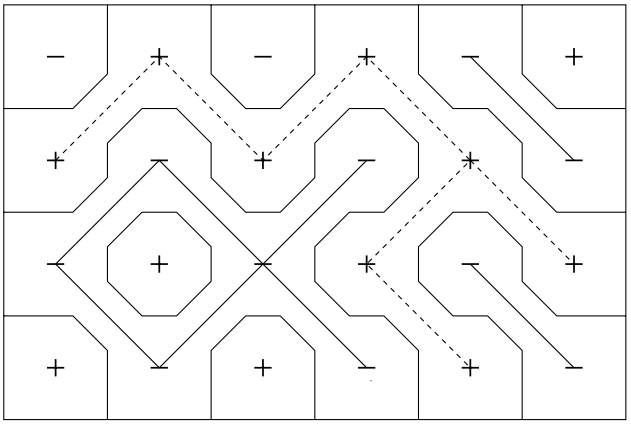

FIG. 3. A realization of random percolationlike process. The plus and minus form two dual lattices. Solid and dashed lines indicate graphs for, respectively, negative and positive dual lattices.

The number of connected nodal domains coincides with the sum of the numbers of different components of both positive and negative graphs. As in [1] first we are interested in the distribution of these numbers. To compute this quantity (unusual for the percolation) it is convenient to connect this model with the Potts model (see, e.g., [8]) similarly as it was done in [9] for a slightly different problem.

Let $n_{\pm}$ be the numbers of connected components of positive and negative graphs. The generating function of their sum is

$$Z(x) = \sum_{\text{realizations}} x^{n_- + n_+}, \tag{4}$$

where variable x plays the role of the fugacity.

The negative and positive graphs, by construction, are dual to each other [10] and their properties are interrelated. In particular (see, e.g., [8], p. 242), $n_+ = C_- + 1$ where C_- is the number of independent circuits on the negative (dual to the positive) graph. According to the Euler relation this quantity can be expressed as follows: $C_- = b_- + n_- - N_-$, where b_- is the number of bonds, n_- is the number of connected components, and N_- is the number of vertices of the negative graph.

These relations permit us to express the generation function (4) through the properties of only negative graph,

$$Z(x) = x^{1-N_s} \sum_{G_-} x^{b_-} (x^2)^{n_-}, \tag{5}$$

where we take into account that N_- equals the total number of sites of negative lattice $N_s = N_{\text{tot}}/2$.

But this quantity is directly connected with the partition sum of the Potts model [8,11]. The later can be defined for an arbitrary graph by the formal sum

$$Z_{\text{Potts}}(v,q) = \sum_G v^{b(G)} q^{n(G)}, \tag{6}$$

where the summation is performed over all graphs G which cover the original graph. $b(G)$ is the number of bonds of this graph, $n(G)$ is its number of connected components. q is the number of states of the Potts model, $v = e^K - 1$ is a parameter related with the inverse temperature K.

Comparing (5) and (6) one gets

$$Z(x) = x^{1-N_s} Z_{\text{Potts}}(x, x^2). \tag{7}$$

The last sum corresponds to the Potts model in the critical point $v^2 = q$; for large rectangular lattice and $q < 4$ it was computed analytically [11]

$$\lim_{N_s \to \infty} \frac{1}{N_s} \ln Z_{\text{Potts}}(x, x^2) = \log x + f(x), \tag{8}$$

where

$$f(x) = \int_{-\infty}^{\infty} \frac{dt}{t} \tanh \mu t \frac{\sinh(\pi - \mu)t}{\sinh \pi t}, \tag{9}$$

and the parameter $\mu(0 < \mu < \pi/2)$ is related to the fugacity x as follows: $\cos\mu = x/2$.

The expansion of expression (7) into series of x gives the number of nodal domains with a fixed number of components N_n. With exponential accuracy

$$\sum_{n=1}^{\infty} N_n x^n = x \exp[N_s f(x)]. \tag{10}$$

We are interested in the behavior of N_n for large n near the maximum of N_n with a fixed number of sites. One has

$$N_n = \frac{1}{2\pi i} \oint \frac{dz}{z^n} \exp[N_s f(z)], \tag{11}$$

where the integration is performed over a contour around zero. Assuming that n is large and using the saddle point method one obtains $N_n \propto \exp\Phi(n, z_c)$, where $\Phi(n, z) = N_s f(z) - n \ln z$ and the saddle point z_c is determined from the equation $\partial\Phi(n, z)/\partial z|_{z=z_c} = 0$. The maximum of N_n corresponds to $n = \bar{n}$ for which $\Phi(n, z_c)$ is maximal. Simple calculation shows that it appears when $z_c = 1$ and

$$\frac{\bar{n}}{N_s} = z \frac{df(z)}{dz}\bigg|_{z=1}. \tag{12}$$

Expanding $\Phi(n, z_c)$ near $n = \bar{n}$ up to the second order and computing all necessary integrals (the details will be given elsewhere [5]), one finds that the total number of nodal domains in the lattice with $N_{\text{tot}} = 2\bar{N}(E)/\pi$ sites [cf. (3)] near the maximum has Gaussian distribution where the mean number of nodal domains $\bar{n}(E)$ and their variance $\sigma^2(E)$ are proportional to $\bar{N}(E)$

$$\frac{\bar{n}(E)}{\bar{N}(E)} = \frac{3\sqrt{3} - 5}{\pi} \approx 0.0624, \tag{13}$$

$$\frac{\sigma^2(E)}{\bar{N}(E)} = \frac{18}{\pi^2} + \frac{4\sqrt{3}}{\pi} - \frac{25}{2\pi} \approx 0.0502. \tag{14}$$

These relations are the main analytical result of this note. They demonstrate that the distribution of nodal domains for random functions is universal (Gaussian) and depends only on the mean number of levels exactly as it is conjectured in [1] with explicitly calculable constants.

In Fig. 4 we present the comparison between the numerical calculation of the mean value and the variance for the random functions (1) for different values of k plotted as functions of $N = \bar{N}(E)$. The dots and squares represent the ratios $\bar{n}(N)/N$ and $\sigma^2(N)/N$ averaged over 100 realizations. The smooth line passing through the dots is the best fit in the form $A + B/\sqrt{N}$. The second term corresponds to a contribution of boundary domains which need special treatment (see [1,10]) but are unessential in the semiclassical limit $N \to \infty$. Within statistical errors numerical results agree well with the asymptotic predictions (13) and (14) [12].

The close relation of our model with the bond percolation model at the critical point $p = 1/2$ permits us to

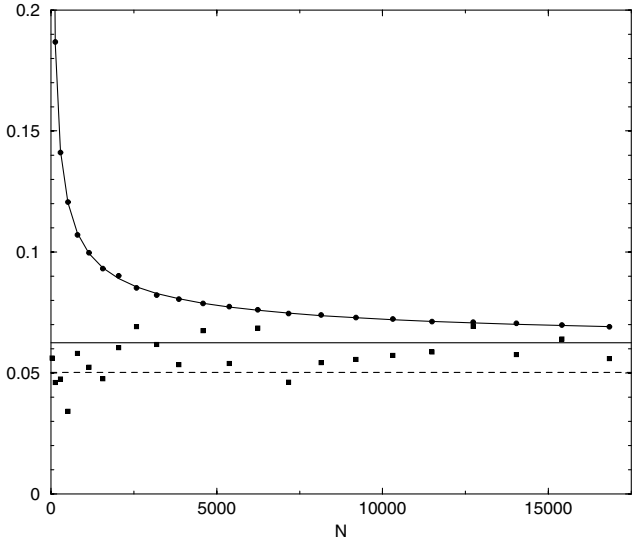

FIG. 4. Mean values of the nodal domains (dots) and their variances (squares) for random functions divided by N versus N. The solid and dashed horizontal lines represent theoretical asymptotic values (13) and (14), respectively.

apply all results of percolation theory (see, e.g., [7] and references therein) for the description of nodal domains of random functions.

In particular, the percolation theory predicts that the distribution of the areas s of clusters (= connected nodal domains) $n(s)$ should have power behavior

$$n(s) \propto s^{-\tau}, \tag{15}$$

where the Fisher exponent $\tau = 187/91$ [7], p. 52.

In Fig. 5 we present the results of numerical calculations of this quantity for random functions (1) with $k = 115$ which are in a rather good agreement with the percolation theory prediction. In this figure the y axis represents the number of nodal domains divided by $\bar{n}(E)$ as in (13). The

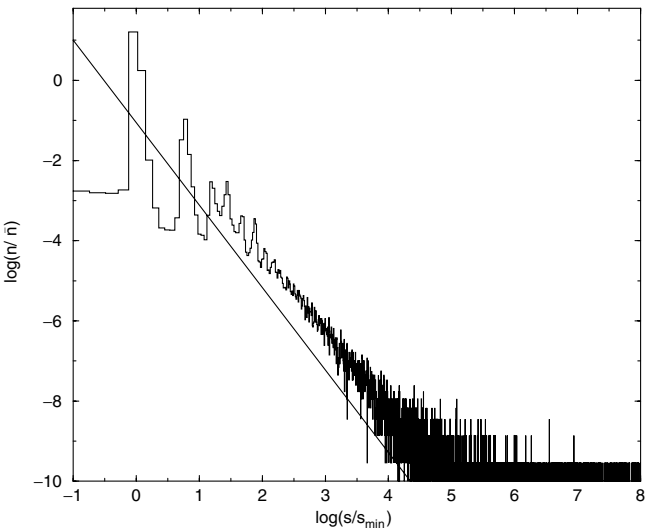

FIG. 5. Distribution of nodal domain areas. The solid line has the slope $\tau = 187/91$ predicted by the percolation theory.

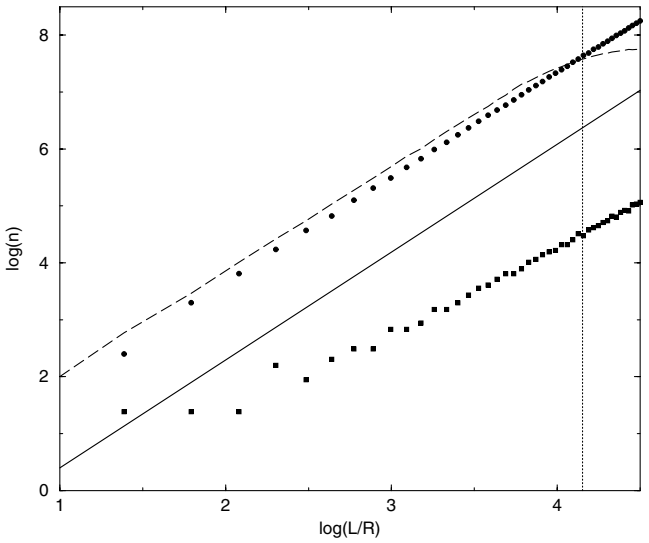

FIG. 6. The number of intersections of nodal domains with square grid of size R. The dotted vertical line indicates the mean lattice size. Circles and squares correspond, respectively, to the big and small highlighted domains in Fig. 1. The dashed line is the result of numerical calculations for the largest cluster in our percolationlike model. The solid line shows percolation theory prediction with the exponent $D = 91/48$.

area along the x axis is measured in the unit of $s_{\min} = \pi(j_1/k)^2$, where j_1 is the first zero of the Bessel function, $J_0(j_1) = 0$, which according to the Rayleigh inequality [13] is the smallest possible (with fixed k) area. After such scaling the results for random functions with different k are practically superimposed. The existence of a discrete set of the smallest possible areas leads to pronounced oscillations at small s in Fig. 5.

Another interesting quantity is the fractal dimension of the nodal domains. In our percolation model it coincides with the fractal dimension of critical percolation clusters which is known to be equal to $D = 91/48$ [7], p. 52.

To find the numerically fractal dimension of a domain it is convenient to put it on a grid of squares of side R and count the number of crossing of the region with the grid. When $a \ll R \ll l$ where l is the size of the domain and a is the size of the mean lattice: $a = 2\pi/k$, one expects

$$n \propto R^{-D} \tag{16}$$

and the exponent D is the fractal dimension.

In Fig. 6 we present numerical verification of this relation for the two nodal domains with $k = 100$ highlighted in Fig. 1 and for the largest cluster in the proposed percolationlike model with the number of sites given by (3). It is clearly seen that the fractal dimension of both domains agrees well with numerical simulations in our percolationlike model and the percolation theory prediction.

To summarize, we developed a simple percolationlike model to describe the nodal domains for random functions. Its main advantage is that all relevant quantities can be computed analytically. By using the relations with the Potts model we demonstrated that the number of nodal domains has Gaussian distribution whose mean value and variance are proportional to the mean staircase function with explicitly calculated parameters (13) and (14). Our results clearly indicate that the distribution of nodal domains for random functions is in the same universality class as critical bond percolation which permits us to predict different critical exponents such as the Fisher exponent for the distribution of the nodal domain areas (15) and its fractal dimension (16).

Many different generalizations of the model considered are possible. We mention only the possibility to use the noncritical percolation model for the description of level domains of random functions, $\Psi(x, y) = \epsilon$, with $\epsilon \neq 0$.

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