

Spontaneous Symmetry Breaking with Abnormal Number of Nambu-Goldstone Bosons and Kaon Condensate

V. A. Miransky*

Department of Applied Mathematics, University of Western Ontario, London, Ontario, Canada N6A 5B7

I. A. Shovkovy*

School of Physics and Astronomy, University of Minnesota, Minneapolis, Minnesota 55455

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We describe a class of relativistic models incorporating a finite density of matter in which spontaneous breakdown of continuous symmetries leads to a lesser number of Nambu-Goldstone bosons than that required by the Goldstone theorem. This class, in particular, describes the dynamics of the kaon condensate in the color-flavor locked phase of high density QCD. We describe the spectrum of low energy excitations in this dynamics and show that, despite the presence of a condensate and gapless excitations, this system is not a superfluid.

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The Goldstone theorem is a cornerstone of the phenomenon of spontaneous breakdown of continuous global symmetries. It is applicable both to relativistic field theories with exact Lorentz symmetry [1] and to most condensed matter systems [2] where this symmetry is absent. However, there is an important difference between these two cases. While in Lorentz invariant systems, the Goldstone theorem is universally valid, it is not so in condensed matter systems. For example, it does not apply to condensed matter systems with long range interactions [2]. From the technical viewpoint, the difference is connected with a kinetic term, and derivative terms, in general, in a Lagrangian density: while their form is severely restricted by the Lorentz symmetry, it is much more flexible in systems where this symmetry is absent.

In this Letter, we describe the phenomenon of spontaneous symmetry breaking of continuous symmetries with an abnormal number of Nambu-Goldstone (NG) bosons taking place at a sufficiently high density of matter in a class of models *without* long range interactions. Here by “abnormal” we understand that the number of gapless NG bosons is less than the number of the generators in the coset space G/H , where G is a symmetry of the action and H is a symmetry of the ground state. On the other hand, as we shall see below, the degeneracy of the ground state remains conventional: it is described by transformations connected with all the generators from the coset space. It is noticeable that this class of models describes a recently suggested [3–5] scenario with a kaon condensate in the color-flavor locked (CFL) phase of high density QCD [6].

We illustrate this phenomenon in a toy model with the following Lagrangian density:

$$\mathcal{L} = (\partial_0 + i\mu)\Phi^\dagger(\partial_0 - i\mu)\Phi - v^2\partial_i\Phi^\dagger\partial_i\Phi - m^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2, \quad (1)$$

where Φ is a complex doublet field and v is a velocity parameter. Since here the Lorentz symmetry is broken by the terms with the chemical potential μ , the velocity

$v \leq 1$ in general. The chemical potential μ is provided by external conditions (to be specific, we take $\mu > 0$) [7]. The above Lagrangian density is invariant under global $SU(2) \times U(1)$. The $SU(2)$ is treated as the isospin group I and the $U(1)$ is associated with hypercharge Y . The electric charge is $Q = I_3 + Y$. This model describes the essence of the dynamics of the kaon condensate [4] (see below).

When $\mu < m$, it is straightforward to derive the tree level spectrum of the physical degrees of freedom. To this end, we switch to the momentum space by decomposing all four real components of the Φ field in plane waves. Then, the quadratic part of the above Lagrangian density takes the following form:

$$\mathcal{L}^{(2)}(\omega, q) = \frac{1}{2} \begin{pmatrix} \phi_1^* & \phi_2^* \end{pmatrix} \mathcal{M} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \tilde{\phi}_1^* & \tilde{\phi}_2^* \end{pmatrix} \tilde{\mathcal{M}} \begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{pmatrix}, \quad (2)$$

where the real and imaginary parts of each component of the doublet were introduced, $\Phi^T = 1/\sqrt{2}(\phi_1 + i\phi_2, \tilde{\phi}_1 + i\tilde{\phi}_2)$. Note that their Fourier transforms satisfy $\phi_i^*(\omega, \vec{k}) = \phi_i(-\omega, -\vec{k})$ and $\tilde{\phi}_i^*(\omega, \vec{k}) = \tilde{\phi}_i(-\omega, -\vec{k})$. The matrices \mathcal{M} and $\tilde{\mathcal{M}}$ in Eq. (2) read

$$\begin{pmatrix} \omega^2 + \mu^2 - m^2 - v^2q^2 & 2i\mu\omega \\ -2i\mu\omega & \omega^2 + \mu^2 - m^2 - v^2q^2 \end{pmatrix}. \quad (3)$$

The dispersion relations of the particles are determined from the equation $\text{Det}(\mathcal{M}) = 0$. Explicitly, this equation reads

$$[(\omega - \mu)^2 - m^2 - v^2q^2] \times [(\omega + \mu)^2 - m^2 - v^2q^2] = 0; \quad (4)$$

i.e., the particle’s dispersion relations are

$$\omega_1 = \tilde{\omega}_1 = \pm(\sqrt{m^2 + v^2q^2} + \mu), \quad (5)$$

$$\omega_2 = \tilde{\omega}_2 = \pm(\sqrt{m^2 + v^2q^2} - \mu). \quad (6)$$

Of course, the positive and negative values of energy correspond to creation and annihilation of excitations, respectively. Henceforth we consider only positive eigenvalues: for our purposes, no additional information is contained in the eigenstates with negative eigenvalues.

Equations (5) and (6) imply that the particle spectrum contains two doublets with the energy gaps $m + \mu$ and $m - \mu$, respectively. In dense quark matter, the first doublet can be identified with (K^-, \bar{K}^0) and the second one with (K^+, K^0) . It is important to note that the chemical potential causes splitting of the masses of particles and their antiparticles. This point is crucial for reducing the number of NG bosons in the asymmetric phase considered below. The splitting is intimately connected with the fact that C , CP , and CPT symmetries are explicitly broken in this system. In second quantized theory, the complex field Φ describes creation and annihilation of (K^-, \bar{K}^0) and (K^+, K^0) , respectively. This is quite unusual because the corresponding dispersion relations of these two doublets are not identical.

By studying the potential of the above model in tree approximation, we could check that the perturbative ground state becomes a local maximum when $\mu > m$ [see Eq. (2) with $\omega = q = 0$]. At this point, the system experiences an instability with respect to forming a condensate. In the new phase, a vacuum expectation value, φ_0 , of the field Φ occurs:

$$\Phi = \begin{pmatrix} 0 \\ \varphi_0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \tilde{\phi}_1 + i\tilde{\phi}_2 \end{pmatrix}. \quad (7)$$

$$\tilde{\omega}_{1,2} = \sqrt{3\mu^2 - m^2 + v^2q^2} \pm \sqrt{(3\mu^2 - m^2)^2 + 4\mu^2v^2q^2}. \quad (12)$$

All four dispersion relations are shown in Fig. 1. As is easy to check, one of the charged states with the relation

$$\omega_2 = \sqrt{\mu^2 + v^2q^2} - \mu \quad (13)$$

and one of the neutral states with the relation

$$\tilde{\omega}_2 = \sqrt{3\mu^2 - m^2 + v^2q^2} - \sqrt{(3\mu^2 - m^2)^2 + 4\mu^2v^2q^2}, \quad (14)$$

describe NG bosons, i.e., gapless excitations whose energy goes to zero as $q \rightarrow 0$ (see Fig. 1). Indeed, in the far infrared region, these relations take the following form:

$$\omega_2 \simeq \frac{v^2q^2}{2\mu}, \quad (15)$$

$$\tilde{\omega}_2 \simeq \sqrt{\frac{\mu^2 - m^2}{3\mu^2 - m^2}} vq. \quad (16)$$

No other gapless states appear. Thus, there are two gapless NG bosons in the system. It comes as a real surprise. Indeed, since the initial global $SU(2) \times U(1)$ symmetry spontaneously breaks down to $U(1)_Q$, one should expect the existence of three NG bosons. Where is the third one? To better understand the situation at hand, it is instructive to consider the quadratic part of the potential around the ground state configuration in the broken phase.

This choice of the “neutral” direction of the vacuum expectation value corresponds to the conventional definition of the electric charge, $Q = I_3 + Y$.

By requiring that the new ground state is a minimum of the potential, we derive

$$\varphi_0^2 = \frac{\mu^2 - m^2}{2\lambda} \quad (8)$$

for the vacuum expectation value of the field. In this ground state, the initial $SU(2) \times U(1)$ spontaneously breaks down to $U(1)_Q$.

In the broken phase, the quadratic part of the Lagrangian density looks formally the same as in Eq. (2). The matrices \mathcal{M} and $\tilde{\mathcal{M}}$, however, are different. In particular, the first one, describing two charged states, is

$$\mathcal{M} = \begin{pmatrix} \omega^2 - v^2q^2 & 2i\mu\omega \\ -2i\mu\omega & \omega^2 - v^2q^2 \end{pmatrix}, \quad (9)$$

while the other, describing two neutral states, is

$$\tilde{\mathcal{M}} = \begin{pmatrix} \omega^2 - 2(\mu^2 - m^2) - v^2q^2 & 2i\mu\omega \\ -2i\mu\omega & \omega^2 - v^2q^2 \end{pmatrix}. \quad (10)$$

For the charged states, the dispersion relations are

$$\omega_{1,2} = \sqrt{\mu^2 + v^2q^2} \pm \mu. \quad (11)$$

For the neutral states, the dispersion relations are given by

For the two sets of the fields, the corresponding quadratic forms are read from $-\mathcal{M}$ and $-\tilde{\mathcal{M}}$, where $\omega = q = 0$ is substituted. The eigenvalues of such matrices are

$$\xi_{1,2} = \tilde{\xi}_2 = 0, \quad (17)$$

$$\tilde{\xi}_1 = 2(\mu^2 - m^2). \quad (18)$$

Therefore, we see that the potential part of the action has three flat directions in the broken phase, as it should. These three directions would correspond to three NG bosons, related to the three broken generators of the original $SU(2) \times U(1)$. The first order derivative terms in the kinetic part of the action, however, prevent the appearance of one charged, K^- , gapless mode. This fact is intimately connected with the point that the presence of the chemical potential leads to splitting of the energy spectra of K^- and K^+ .

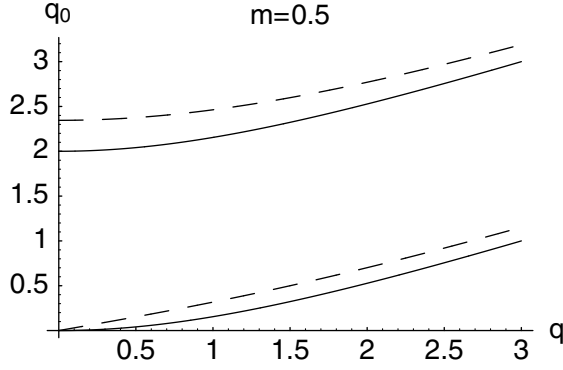


FIG. 1. Dispersion relations of four particles in the broken phase of the model in Eq. (1). The solid and dashed lines represent the dispersion relations of charged and neutral states, respectively. All quantities are given in units of μ . To plot the figure we used $m = 0.5\mu$ and $v = 1/\sqrt{3}$.

Another noticeable fact is that the energy ω_2 of the K^+ gapless mode in Eq. (15) is proportional to q^2 rather than to q . This implies that, despite the presence of the condensate, the Landau criterion for superfluidity fails in this model [8]. Therefore, this system is not a superfluid.

This toy model illustrates a rather general phenomenon. While in Lorentz invariant systems with spontaneous breakdown of continuous symmetries the degeneracy of the potential fixes the number of NG bosons being equal to the number of generators $N_{G/H}$ in the coset space G/H , in systems with a broken Lorentz symmetry the number of gapless NG bosons can be lesser. The latter is connected with a form of the kinetic term which can include first order derivative terms. In the systems under consideration, the chemical potential (leading here to such first order derivative terms) splits up energy gaps of charged particles and their antiparticles *both* of which would be NG bosons otherwise. On the other hand, a neutral NG boson, a partner of a field with a nonzero vacuum expectation value, always survives. Therefore, the number of the gapless NG bosons reduces by N_{ch} , where N_{ch} is the number of charged (particle-antiparticle) pairs among the would-be NG bosons. In the present model $N_{G/H} = 3$ and $N_{\text{ch}} = 1$, and as a result there are only two (one neutral and one charged) NG bosons.

The choice of the simple model in Eq. (1) was not accidental in this Letter. It describes the essence of a much more complicated dynamics of spontaneous symmetry breaking related to the kaon condensation in the color-flavor locked phase of dense quark matter [4]. Let us turn to it.

We start by repeating briefly the analysis of Ref. [4]. In the chiral limit, the ground state of the three flavor dense quark matter corresponds to the CFL phase [6]. The original $SU(3)_c \times SU(3)_L \times SU(3)_R$ symmetry of the microscopic action breaks down to the global “locked” $SU(3)_{c+L+R}$ subgroup. The corresponding low energy action for the NG bosons was derived in Refs. [9,10]. The current quark masses break the original chiral symmetry of

the model explicitly. As a result, nonzero gaps appear in the spectra of the NG bosons, and they become pseudo-NG bosons. Their dynamics could still be described by the low energy action which, for sufficiently small current quark masses, could be derived from the microscopic theory [10]. By making use of an auxiliary “gauge” symmetry, it was suggested in Ref. [4] that the low energy action of Refs. [9,10] should be modified by adding a timelike covariant derivative to the action of the composite field.

By neglecting a chemical potential of the electric charge, the low energy effective Lagrangian density of Ref. [4] (in Minkowski space) reads

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & \frac{f_\pi^2}{4} \text{Tr}[\nabla_0 \Sigma \nabla_0 \Sigma^\dagger - v_\pi^2 \partial_i \Sigma \partial_i \Sigma^\dagger] \\ & + \frac{1}{2} [(\partial_0 \eta')^2 - v_\eta^2 (\partial_i \eta')^2] \\ & + 2c[\det(M) \text{Tr}(M^{-1} \Sigma e^{\sqrt{2/3}(i/f_\eta)I \eta'}) + \text{H.c.}], \end{aligned} \quad (19)$$

$$\nabla_0 \Sigma = \partial_0 \Sigma + i \frac{MM^\dagger}{2p_F} \Sigma - i \Sigma \frac{M^\dagger M}{2p_F}, \quad (20)$$

where p_F is the quark Fermi momentum and M is a quark mass matrix chosen to be diagonal, i.e., $M = \text{diag}(m_u, m_d, m_s)$. By definition, I is a unit matrix in the flavor space, and Σ is a unitary matrix field which describes the octet of the NG bosons, transforming under the chiral $SU(3)_L \times SU(3)_R$ group as follows:

$$\Sigma \rightarrow U_L \Sigma U_R^\dagger, \quad (21)$$

where $(U_L, U_R) \in SU(3)_L \times SU(3)_R$. In Eq. (19), we also took into account η' field which couples to the octet when the quark masses are nonzero. The NG boson, related to breaking the baryon number, was omitted, however. Its dynamics is not affected much by the quark masses.

One should notice from the definition of the covariant derivative in Eq. (20) that the combination of the quark mass matrices $\mu_{\text{eff}} = MM^\dagger/2p_F$ produces effective chemical potentials for different flavor charges. These chemical potentials are of dynamical origin, and they are unavoidable.

The presence of the effective chemical potential μ_{eff} has far-reaching consequences. In particular, if the mismatch of the quark masses of different flavors is large enough [e.g., $m_s \gtrsim (\Delta^2 m_u)^{1/3}$], the perturbative CFL ground state becomes unstable with respect to a kaon condensation.

The new ground state is determined by a “rotated” vacuum expectation value of the Σ field,

$$\begin{aligned} \Sigma_\alpha(\pi) & \equiv \exp(i\alpha\lambda^6) \exp(i\pi_A \lambda^A) \\ & \simeq \exp(i\alpha\lambda^6) \\ & \quad \times \left(1 + \frac{i\pi_A \lambda^A}{f_\pi} - \frac{\pi_A \pi_B \lambda^A \lambda^B}{2f_\pi^2} + \dots \right), \end{aligned} \quad (22)$$

where α is determined by requiring that the corresponding ground state configuration is a global minimum of the

potential energy. In the simplest case with $m_u = m_d$, we derive

$$\cos\alpha = \frac{4c p_F^2 m_u (m_s + m_u)}{f_\pi^2 (m_s^2 - m_u^2)^2} < 1, \quad (23)$$

where (see Ref. [10])

$$c = \frac{3\Delta^2}{2\pi^2} \quad \text{and} \quad f_\pi^2 = \frac{21 - 8\ln 2}{36} \frac{p_F^2}{\pi^2}. \quad (24)$$

The ground state with the kaon condensation which is determined by Eq. (22) breaks the $SU(2) \times U(1)_Y$ symmetry of the effective action (19) down to $U(1)_Q$. This is exactly the symmetry breaking pattern that we encountered in the toy model.

The derivation of the dispersion relations in the broken phase with the kaon condensation involves rather tedious calculations. Qualitatively, though, such a derivation is similar to that in the model in Eq. (1). Additional difficulties come from more complicated particle mixing. By omitting the details, we present the results.

With the choice of the vacuum expectation value in Eq. (22), the original 9 degrees of freedom (π_A with $A = 1, \dots, 8$ and η') group into two decoupled sets: $(\pi_1, \pi_2, \pi_4, \pi_5)$ and $(\pi_3, \pi_6, \pi_7, \pi_8 \equiv \eta, \eta')$. The first set of states contains all charged degrees of freedom (i.e., π^\pm and K^\pm), while the second set contains neutral ones (i.e., $\pi^0, K^0, \bar{K}^0, \eta$, and η').

The dispersion relations for the charged degrees of freedom are straightforward to derive. Our analysis shows that there is only one gapless NG boson in this set. In the far infrared region $q \rightarrow 0$, its explicit dispersion relation reads

$$\omega \simeq \frac{q^2}{3\mu \cos\alpha(1 + \cos\alpha)}, \quad (25)$$

where $\mu = (m_s^2 - m_u^2)/2p_F$. The analysis of the dispersion relations of the other set is more difficult. But it is also straightforward to show that there is only one gapless NG boson there as well. Moreover, for $q \rightarrow 0$, the explicit dispersion relation reads

$$\omega \simeq \frac{\sin\alpha}{\sqrt{2 - 7\cos^2\alpha + 9\cos^4\alpha}} \frac{q}{\sqrt{3}}. \quad (26)$$

We see that, as in the case of the simple model in Eq. (1), there are only two gapless NG bosons, despite the fact that there are three broken symmetry generators in the phase with the kaon condensate. One should note, however, that as in the model in Eq. (1), the effective potential obtained from (19) has the required three flat directions. It is the first order derivative terms that are responsible for producing a nonzero energy gap in the spectrum of one of the charged bosons. The corresponding opposite charge partner remains a gapless NG boson. It is crucial to note,

though, that its dispersion relation behaves as $\omega \sim q^2$ for $q \rightarrow 0$. This implies that the criterion of superfluidity is not satisfied in the phase of the dense quark matter with kaon condensation [8].

As has been argued in Refs. [4,5], there is a good chance that the phase with a kaon condensate may exist in a core of compact stars. Since the spectrum of low energy excitations in this phase derived in the present Letter is very specific, implying in particular that the matter is not superfluid, it could play an important role in detecting this phase.

In conclusion, in this Letter the phenomenon of spontaneous symmetry breaking with an abnormal number of NG bosons was described. It admits a simple and clear interpretation. We expect that there exist wide applications of this phenomenon that deserve further study. In passing, we note that related systems in condensed matter physics are ferromagnets [11] and the superfluid ^3He in the so-called A phase [12].

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*On leave of absence from Bogolyubov Institute for Theoretical Physics, 252143, Kiev, Ukraine.

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