## **Spontaneously Broken Spacetime Symmetries and Goldstone's Theorem**

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Goldstone's theorem states that there is a massless mode for each broken symmetry generator. It has been known for a long time that the naive generalization of this counting fails to give the correct number of massless modes for spontaneously broken spacetime symmetries. We explain how to get the right count of massless modes in the general case, and discuss examples involving spontaneously broken Poincaré and conformal invariance.

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The proof of Goldstone's theorem for internal symmetries is now standard material in many textbooks on quantum field theory. Briefly stated, the theorem asserts that, for a physical system with a global internal symmetry group G which is spontaneously broken down to a subgroup H, there is a massless mode corresponding to each broken generator [1]. In other words, the number of Goldstone bosons is equal to the dimension  $\dim(G/H) =$  $\dim(G) - \dim(H)$  of the coset space G/H. (For the purposes of this paper, we will ignore subtleties such as the Coleman-Mermin-Wagner theorem on Goldstone bosons in two dimensions.) Moreover, if the global symmetry Gis gauged into a local symmetry group, some of the gauge bosons become massive through the Higgs mechanism. In this case, the number of massive bosons, which equals the number of would-be Goldstone bosons, is still  $\dim(G/H)$ .

The naive generalization of this counting fails to give the correct number of massless modes for spontaneously broken spacetime symmetries. There are at least two well-known cases. One is the spontaneous breaking of rotational and translational invariance due to, for example, an extended object such as a domain wall or D-brane. In examples discussed in Refs. [2,3], there are massless modes corresponding to only the broken translational generators. The second example is the spontaneous breaking of conformal symmetry down to Poincaré symmetry. In four dimensions, the conformal group has 15 generators, whereas the Poincaré group has 10 generators. Naive counting using  $\dim(G) - \dim(H)$  would give five Goldstone modes. However, as was discussed in Refs. [4,5], there is actually only one massless mode, corresponding to the dilatation generator.

It is known that, in non-Lorentz-invariant theories, the number of massless modes can be less than the number of broken generators even for internal symmetries, and the rule for counting the massless modes is given in Ref. [6]. We analyze a different problem—that of counting massless modes in Lorentz-invariant theories with broken spacetime symmetries. Following a review of the two well-known examples of spontaneously broken Poincaré and conformal symmetries, we present the criterion for counting massless modes, and show how the same result can be derived by applying the coset construction of spontaneously broken symmetries [5,7].

A large class of models with extra dimensions consider quantum fields confined in a (p + 1) dimensional hypersurface, which is generally called a *p*-brane, embedded in a *d*-dimensional spacetime, with d > p. If the vacuum state of a theory in *d*-dimensional flat space contains a *p*-brane, the *d*-dimensional Poincaré group, denoted by ISO(d - 1, 1), is spontaneously broken down to the (p + 1)-dimensional Poincaré group ISO(p, 1). [We use the Minkowski metric  $\eta_{MN} = (-, +, +, ...)$ .] We will consider this pattern of symmetry breaking; the spontaneous breaking of Poincaré invariance by vortices or domain walls are special cases.

The indices of the bulk spacetime will be denoted by capital italic letters,  $M, N = 0, \dots, d - 1$ , the *p*-brane coordinate indices by Greek letters,  $\mu, \nu = 0, \dots, p$ , and the remaining d - (p + 1) of the bulk indices by lowercase italic letters m, n = p + 1, ..., d - 1. The coordinates of the bulk spacetime are  $X^M$ , and those intrinsic to the *p*-brane are  $x^{\mu}$ . The translation generators  $P^{M}$  can be divided into two sets,  $P^{\mu}$ , which remain unbroken, and  $P^{m}$ , which are broken by the *p*-brane. The Lorentz generators  $J^{MN}$  split into the unbroken generators  $J^{\mu\nu}, J^{mn}$  and the broken generators  $J^{\mu n}$ . The position of a point x on the *p*-brane is described by the bulk coordinates  $Y^{M}(x)$ . It is possible to choose a gauge such that  $Y^{\mu}(x) = x^{\mu}$  and the remaining components,  $Y^m(x)$ , can be thought of as the Goldstone modes corresponding to the broken translational generators [3], which describe the fluctuations of the p-brane in the transverse directions. The number of Goldstone modes is d - p - 1, which is the same as the number of broken translation generators  $P^m$ . There are no additional Goldstone modes corresponding to the broken Lorentz generators  $J^{\mu n}$ .

Next consider the spontaneous breaking of conformal symmetry [4]. Any Lagrangian with a symmetry group H can be made G invariant,  $G \supset H$ , by adding Goldstone bosons so that it appears the symmetry G is spontaneously broken. When G is taken to be the conformal group and H the Poincaré group, although the broken generators are the dilatation and special conformal transformations, we only

need one massless mode  $\sigma(x)$ , the dilaton, to make the Lagrangian conformally invariant. (This is related to the fact that a theory which is scale invariant is also conformal invariant if a certain condition is satisfied, which is true for a wide class of theories [8,9].)

As a simple example, consider a scalar  $\phi^4$  theory in four dimensions which can be made conformally invariant by adding the dilaton  $\sigma(x)$  in the following way:

$$S = \int d^{4}x \left[ \frac{1}{2} (\partial_{\mu} + f \partial_{\mu}\sigma)\phi(\partial^{\mu} + f \partial^{\mu}\sigma)\phi + \Lambda e^{-4f\sigma} - \frac{1}{2}m^{2}\phi^{2}e^{-2f\sigma} - \frac{\lambda}{4}\phi^{4} + \frac{1}{2}e^{-2f\sigma}\partial_{\mu}\sigma\partial^{\mu}\sigma \right].$$
(1)

Note that, under a scale transformation  $x \rightarrow e^{-d}x$ , the field  $\sigma$  transforms in a nonlinear way  $\sigma(x) \rightarrow \sigma(e^{-d}x) - d/f$  and is indeed the Goldstone mode corresponding to the dilatation. The Lagrangian Eq. (1) describes a theory with spontaneously broken conformal symmetry, with one massless mode coupling to the dilatation current. There are no additional massless modes corresponding to the breaking of the special conformal transformations.

Assume that a symmetry group G with dim(G) generators  $T^A$  (capital italic superscript) is broken down to a symmetry group H with dim(H) generators  $T^{\alpha}$ (Greek superscript). The remaining dim(G) – dim(H) generators  $T^a$  (lowercase italic superscript) are referred to as the broken generators. Let  $\phi(\mathbf{r})$  be the symmetry breaking order parameter,  $T^{\alpha}\langle \phi(\mathbf{r}) \rangle = 0$ , and  $T^a \langle \phi(\mathbf{r}) \rangle \neq 0$ . In the case of internal symmetry breaking,  $\phi(\mathbf{r})$  is a scalar field, but for spacetime symmetry breaking,  $\phi(\mathbf{r})$  can be a tensor field.

Consider first the case of a broken internal symmetry. The massless modes are small amplitude long-wavelength fluctuations of the order parameter,

$$\delta \phi(\mathbf{r}) = c_A(\mathbf{r}) T^A \langle \phi(\mathbf{r}) \rangle = c_a(\mathbf{r}) T^a \langle \phi(\mathbf{r}) \rangle, \quad (2)$$

where  $c_a(\mathbf{r})$  is now a slowly varying function of  $\mathbf{r}$ . The generators  $T^{\alpha}$  corresponding to the unbroken generators do not generate massless excitations, since  $T^{\alpha} \langle \phi(\mathbf{r}) \rangle = 0$ . The remaining  $c_a$  can be chosen independently, and the number of independent modes is clearly the same as the number of broken generators, dim(G/H).

For spontaneously broken spacetime symmetries, the number of massless modes is no longer equal to the number of broken generators. Massless modes are still given by small amplitude long-wavelength fluctuations of the order parameter [Eq. (2)], where  $c_a$  can depend on the coordinates **r** in the directions in which translation remains unbroken. The number of independent massless modes is the number of broken generators dim(G/H) minus  $n_x$ , the number of independent solutions to

$$c_a(\mathbf{r})T^a\langle\phi(\mathbf{r})\rangle = 0.$$
(3)

The key point is that  $n_x \ge 0$ : there can be nontrivial solutions to Eq. (3) when  $c_a$  and  $T^a$  both depend on **r**.

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The generators  $T^a$  are linearly independent, but the longwavelength fluctuations they produce need not be. The number of Goldstone bosons is then dim $(G/H) - n_x$ , and is reduced from the naive counting of broken generators. Equation (3) can always be used to determine  $n_x$ , even in the case of internal symmetries. If the generators  $T^a$  are internal generators, Eq. (3) has no nontrivial solutions, and  $n_x = 0$ .

It is easy to see how there could be nontrivial solutions to Eq. (3), thus reducing the number of Goldstone modes. Consider in three dimensions, a ground state with an infinitely long, straight string parallel to the y axis, as shown in Fig. 1. The three-dimensional Poincaré group is spontaneously broken to the two-dimensional Poincaré group. A rotation in the x-y plane changes the orientation of the string, whereas a translation in the x direction shifts the string parallel to itself. The effect of these two symmetry operations is apparently very different. Nevertheless, if we perform a *local* translation on the string, in the sense that the amount of translation is different at every point on the string, the effect is to produce a bump on the string. This bump can clearly be compensated by performing a *local* rotation on the string (see Fig. 1).

The translational symmetry breaking of  $P_x$  produces massless modes,

$$\delta \phi(\mathbf{r}) = \epsilon(y) P_x \langle \phi(\mathbf{r}) \rangle,$$
 (4)

where  $\epsilon$  depends only on y, the coordinate in the direction of the unbroken translation generator  $P_y$ . (Goldstone modes can only propagate in the direction of the unbroken translations. They have a dispersion relation  $\omega(k)$  with  $\omega(k) \rightarrow 0$  as  $k \rightarrow 0$ . k is defined only in the translationally invariant directions. The broken translation  $P_x$  generates translational zero mode describing the fluctuations of the string in the x direction.) Similarly, rotational symmetry breaking gives the mode

$$\delta \phi(\mathbf{r}) = \theta(\mathbf{y}) J_{\mathbf{x}\mathbf{y}} \langle \phi(\mathbf{r}) \rangle, \tag{5}$$

where  $\theta$  depends only on y. Requiring these two fluctuations to exactly cancel each other, we have

$$\boldsymbol{\epsilon}(\mathbf{y})P_{\mathbf{x}}\langle\boldsymbol{\phi}(\mathbf{r})\rangle = -\mathbf{y}\boldsymbol{\theta}(\mathbf{y})P_{\mathbf{x}}\langle\boldsymbol{\phi}(\mathbf{r})\rangle, \qquad (6)$$



FIG. 1. A ground state with a string breaks the threedimensional Poincaré group down to the two-dimensional Poincaré group. Global translation and rotation on the string are distinctly different, whereas the effects of local translation and rotation on the string can be made the same.

where we have used  $P_y\langle \phi(\mathbf{r}) \rangle = 0$  and the relation  $J_{xy} = xP_y - yP_x$  valid for spinless particles. [More precisely, one uses the relation  $M^{\mu\nu\lambda}(x) = x^{\nu}T^{\mu\lambda}(x) - x^{\lambda}T^{\mu\nu}(x)$  between the stress tensor and the angular momentum density.] Equation (6) is clearly satisfied by choosing  $\epsilon(y) = -y\theta(y)$ . Note that no solution would be possible if  $\epsilon$  and  $\theta$  were both chosen to be constants. In this example  $P_x$  and  $J_{xy}$  do not generate independent massless excitations, and  $n_x > 0$ .

In the general case, acting on Eq. (3) with the unbroken translation  $P_{\mu}$  gives

$$0 = P_{\mu}c_{a}(\mathbf{r})T^{a}\langle\phi(\mathbf{r})\rangle = [P_{\mu}, c_{a}(\mathbf{r})T^{a}]\langle\phi(\mathbf{r})\rangle$$
$$= -i[\partial_{\mu}c_{a}(\mathbf{r})T^{a} - f^{\mu ab}c_{a}(\mathbf{r})T^{b}]\langle\phi(\mathbf{r})\rangle, \quad (7)$$

where we have written the commutator in the most general form

$$[P_{\mu}, T^a] = i f^{\mu a b} T^b + i f^{\mu a \beta} T^{\beta}.$$
(8)

The  $T^{\beta}$  are unbroken generators and thus annihilate the vacuum. If the  $T^{a}$  are internal generators,  $[P_{\mu}, T^{a}] = 0$ , and Eq. (7) implies that  $c_{a}$  are constant, so that Eq. (3) has no nontrivial solutions.

As long as there are some nonzero  $f^{\mu ab}$ , the nontrivial solution satisfies

$$[\partial_{\mu}c_{a}(\mathbf{r}) - c_{b}(\mathbf{r})f^{\mu ba}]T^{a}\langle\phi(\mathbf{r})\rangle = 0.$$
<sup>(9)</sup>

This is equivalent to saying that the Goldstone mode for  $T^b$  and the gradient of the Goldstone mode for  $T^a$  are linearly dependent, so they do not generate independent massless excitations. The nontrivial solutions to Eq. (9) reduce the number of Goldstone bosons, and there is a one-to-one correspondence between the nontrivial solutions of Eq. (9) and Eq. (3). We will see that Eq. (9) also occurs in the coset construction of the low energy effective theory.

Propagating Goldstone modes exist when there are unbroken translational directions. It is also interesting to consider configurations which break all the translational invariance; for example a soliton such as a magnetic monopole or Skyrmion. In this case, one has zero modes that correspond to changes in the collective coordinates of the soliton. The counting here is the same as in the case of internal symmetry, and there is no relation between the translational and rotational generators. We can see this from Eq. (3) as well. We stressed that the spacetime dependence of the coefficients  $c_a(\mathbf{r})$  is on the coordinate in the unbroken translation. When all the translations are broken,  $c_a(\mathbf{r})$  have no spacetime dependence and are purely constants, Eq. (3) has no nontrivial solutions, and the counting is the same as for internal symmetries.

To see how the counting of Goldstone modes works for the two examples discussed earlier, let us write the full conformal algebra:

$$[J_{MN}, J_{PQ}] = i(\eta_{NP} J_{MQ} - \eta_{MP} J_{NQ} - \eta_{NQ} J_{MP} + \eta_{MQ} J_{NP}), \quad (10)$$

$$[J_{MN}, P_Q] = i(\eta_{NQ}P_M - \eta_{MQ}P_N), \qquad (11)$$

$$[J_{MN}, K_Q] = i(\eta_{NQ}K_M - \eta_{MQ}K_N), \qquad (12)$$

$$[P_M, K_N] = 2iJ_{MN} - 2i\eta_{MN}D, \qquad (13)$$

$$[D, K_M] = iK_M , \qquad (14)$$

$$[D, P_M] = -iP_M, \qquad (15)$$

where *D* is the generator for dilatation, and  $K^M$  are the generators for special conformal transformations. If the *d*-dimensional Poincaré group is broken down to the (p + 1)-dimensional Poincaré group due to the presence of the *p*-brane, we have, from Eq. (11),

$$[P_{\mu}, J_{\nu m}]\langle \phi(\mathbf{r}) \rangle = i \eta_{\mu \nu} P_m \langle \phi(\mathbf{r}) \rangle, \qquad (16)$$

where  $J_{\mu m}$  and  $P_m$  are the broken rotational and translational generators, respectively. Therefore  $J_{\nu m}$  and  $P_m$ do not generate independent Goldstone modes. Similarly, for the conformal group spontaneously broken down to the Poincaré group, we have, from Eq. (13),

$$[P_M, K_N]\langle\phi(\mathbf{r})\rangle = -2i\eta_{MN}D\langle\phi(\mathbf{r})\rangle \qquad (17)$$

and therefore all the modes for  $K_N$  can be eliminated, leaving only the dilaton.

Next we apply the coset construction for theories with spontaneous symmetry breaking, introduced in Ref. [7] for internal symmetries and modified in Ref. [5] for spacetime symmetries, to the discussion of counting massless modes. It is convenient to divide the unbroken generators,  $T^{\alpha}$ , into the unbroken momenta,  $P_{\mu}$ , and the rest,  $V_s$ . Consider the group element

$$\Omega(x,\xi) = e^{ix^{\mu}P_{\mu}}e^{i\xi^{a}(x)T_{a}},$$
(18)

which transforms under the action of an element g of G as

$$ge^{ix^{\mu}P_{\mu}}e^{i\xi^{a}T_{a}} = e^{ix'^{\mu}P_{\mu}}e^{i\xi'^{a}(x')T_{a}}h[\xi^{a}(x),g], \quad (19)$$

where  $h[\xi^a(x), g]$  is an element of H depending on  $\xi^a(x)$ and g. If g belongs to the unbroken group H, the transformation of  $x^{\mu}$  and  $\xi^a(x)$  becomes linear. For example, if g is one of the unbroken Lorentz generators, it simply induces the usual Lorentz transformation  $x' = \Lambda x$ and  $\xi'(x') = S^{-1}(\Lambda)\xi(x)$ . However, under a translation  $e^{iy^{\mu}P_{\mu}}$ , the spacetime coordinates always transform inhomogeneously, x' = x + y, whereas  $\xi'(x') = \xi(x)$ . [Recall that we can think of the spacetime coordinates  $x^M$ as parametrizing the coset (Poincaré)/(Lorentz).] This is why  $P_{\mu}$  play the same role as other broken generators in  $\Omega(x, \xi)$ .

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In order to construct an effective action invariant under the full symmetry G, we need to consider the Maurer-Cartan one form

$$\Omega^{-1}(x,\xi)d\Omega(x,\xi) = i(\omega_P^{\mu}P_{\mu} + \omega_T^a T_a + \omega_V^s V_s).$$
(20)

The one forms  $\omega_P^{\mu}$  and  $\omega_T^a$  transform covariantly and are related to the spacetime vielbeins and the covariant derivatives of the Goldstone field  $\xi_a$ :

$$\omega_P^{\alpha} = dx^{\mu} e_{\mu}^{\alpha}, \qquad (21)$$

$$\omega_T^a = dx^\mu e_\mu^\alpha \,\, \nabla_\alpha \,\, \xi^a. \tag{22}$$

On the other hand,  $\omega_V$  is the gauge field (sometimes called the spin connection) associated with the unbroken group *H*,

$$\omega_V^s = dx^\mu \omega_{V\mu}^s \,, \tag{23}$$

and has the same transformation law as the gauge field under local transformations of H.

In order to compute Eq. (20), we need the following commutation relations, written in the most general form:

$$[P_{\mu}, T_{a}] = i f^{\mu a \nu} P_{\nu} + i f^{\mu a b} T_{b} + i f^{\mu a s} V_{s}, \quad (24)$$

$$[T_a, T_b] = i f^{ab\mu} P_\mu + i f^{abc} T_c + i f^{abs} V_s \,. \tag{25}$$

Therefore  $f^{\mu a\nu}$  and  $f^{ab\mu}$  contribute to the spacetime vielbeins [Eq. (21)],  $f^{\mu ab}$  and  $f^{abc}$  contribute to the covariant derivative of the Goldstone boson [Eq. (22)], and  $f^{\mu as}$  and  $f^{abs}$  contribute to the spin connection [Eq. (23)]. Focusing on the Goldstone field, and working at linearized order, we have

$$\omega_T^a = (\partial_\mu \xi^a - f^{\mu b a} \xi^b) dx^\mu.$$
(26)

The effective Lagrangian contains  $\Omega^{-1}d\Omega$  acting on  $\langle \phi \rangle$ , so that the Goldstone boson fields occur via

$$\omega_T^a T^a \langle \phi \rangle = (\partial_\mu \xi^a - f^{\mu b a} \xi^b) dx^\mu T^a \langle \phi \rangle.$$
 (27)

Here we see the possibility of expressing some of the Goldstone modes in terms of derivative of other Goldstone modes by setting Eq. (27) to zero, which reduces the number of independent Goldstone modes that occur in the effective Lagrangian. Note that the linearized covariant derivative is exactly Eq. (9), the condition for nontrivial solutions to Eq. (3).

For the case of a *p*-brane breaking the Poincaré group spontaneously, we chose to write  $\Omega$  as

$$\Omega = e^{ix^{\mu}P_{\mu}}e^{iY^a(x)P_a}e^{i\theta^{\nu b}(x)J_{\nu b}}.$$
(28)

The covariant derivative of the Goldstone mode  $Y^{a}(x)$  is

$$\omega_P^b = \left[ R(\theta)^b_\mu + R(\theta)^b_a \partial_\mu Y^a \right] dx^\mu, \qquad (29)$$

where

$$[R(\theta)]_{MN} = (e^{i\theta^{\nu b} \Sigma_{\nu b}})_{MN}, \qquad (30)$$

$$(\Sigma^{\nu b})_{MN} = i(\delta^{\nu}_{M}\delta^{b}_{N} - \delta^{\nu}_{N}\delta^{b}_{M}).$$
(31)

The covariant derivative of the Goldstone field  $Y^a(x)$  involves  $Y^a$  as well as  $\theta^{\nu b}(x)$ , the Goldstone field for the broken rotational generators. It is therefore possible to solve for  $\theta^{\nu b}(x)$  in terms of the derivatives of  $Y^a(x)$  by setting the covariant derivative to zero.

For the spontaneous breaking of conformal symmetry, we follow Ref. [5] and write

$$\Omega = e^{ix \cdot P} e^{i\varphi(x) \cdot K} e^{i\sigma(x)D}.$$
(32)

The covariant derivative of the dilaton is

$$\omega_D = (\partial_\mu \sigma + 2\varphi_\mu) dx^\mu. \tag{33}$$

Again we can replace the field  $\varphi_{\mu}$  everywhere by  $-(1/2)\partial_{\mu}\sigma$  by setting the covariant derivative of the dilaton field to zero. The fact that one can eliminate some Goldstone fields this way is called the inverse Higgs effect in Ref. [10].

As a final note, it should be clear that choosing a different parametrization of the coset space G/H would give a different relation among the various Goldstone modes. Nevertheless, the number of massless modes is determined by the nonvanishing  $f^{\mu ab}$  in Eq. (24), and the number of Goldstone modes is independent of the parametrization of the coset.

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