Variational Principles and Self-Organization in Two-Fluid Plasmas

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Self-organization of an ordered structure occurs in a plasma under rather restrictive conditions. A new framework for a variational principle invokes a coercive form that results in a criterion for self-organizing relaxation of a two-fluid plasma. The constraints (constants of motion of the ideal model) are adjusted, through a weakly dissipative process, so that the relaxed state, under well-defined conditions, is a stable equilibrium independent of the direct effects of dissipation.

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We critically examine the phenomenon of self-organization of plasmas into "relaxed states" of varying complexity. Since many of the relaxed states in recent literature follow from seemingly standard variational principles, one could draw the erroneous conclusion that self-organization is a general tendency of plasmas. We presently show that it is not so; self-organization, in fact, may occur only under rather restrictive conditions—not all variational principles are well posed, and even when they are, not all solutions to variational principles lead to stable equilibria (even to equilibria)—an essential minimum qualification for relaxed states.

Stability of a state may be proved if the kinetic part of an appropriate total energy can be shown to be bounded. If the "energy" (a constant of motion) can be split into well-defined kinetic and "potential" parts, the state with the minimum potential energy is guaranteed to be stable. In this sense, the variational principle (VP1) and the relaxed states ($\nabla \times \boldsymbol{B} = \lambda \boldsymbol{B}$) derived by Taylor [1] by minimizing the magnetic energy $E_m = \int |\boldsymbol{B}|^2\, dx$ keeping the magnetic helicity $H_1 = \int \boldsymbol{A} \cdot \boldsymbol{B}\, dx$ constant readily pass the stability muster; for sufficiently small $|\lambda|$, any departure from the Taylor state must increase the magnetic energy, resulting in a decrease of the remaining kinetic part because the sum of the kinetic and magnetic energies is a constant in the framework of the incompressible single fluid magnetohydrodynamics (MHD). The standard "energy principle" for a static MHD equilibrium (steady state without flow) invokes a more complicated general expression of the potential energy. For an equilibrium with a stationary flow, however, the generator of the linearized system is non-Hermitian and the energy of the perturbations is no longer well defined. Hence, the situation is less satisfactory for the variational principle (VP2) [2] used to derive the more general relaxed states pertinent to a two-fluid plasma in which the combination of the velocity and the magnetic field control the full dynamics.

Before discussing the variational principle VP2, we summarize some basic properties of the magnetofluid system expressible in the succinct vortex dynamical form:

$$\frac{\partial}{\partial t}\boldsymbol{\omega}_j - \nabla \times (\boldsymbol{U}_j \times \boldsymbol{\omega}_j) = 0 \qquad (j = 1, 2) \tag{1}$$

in terms of a pair of generalized vorticities and the corresponding flows

$$\begin{cases} \boldsymbol{\omega}_1 = \boldsymbol{B}\,, \\ \boldsymbol{U}_1 = \boldsymbol{V} - \nabla \times \boldsymbol{B}\,, \end{cases} \qquad \begin{cases} \boldsymbol{\omega}_2 = \boldsymbol{B} + \nabla \times \boldsymbol{V}\,, \\ \boldsymbol{U}_2 = \boldsymbol{V}\,. \end{cases}$$

The notation is standard with the velocity, the magnetic field, and the length, respectively, normalized to the Alfvén speed, to a characteristic magnetic field, and the ion collisionless skin depth. The three integral invariants

$$E = \int_\Omega (|\boldsymbol{V}|^2 + |\boldsymbol{B}|^2)\, dx\,, \tag{2}$$

$$H_1 = \int_\Omega \boldsymbol{A} \cdot \boldsymbol{B}\, dx\,, \tag{3}$$

$$H_2 = \int_\Omega (\boldsymbol{A} + \boldsymbol{V}) \cdot (\boldsymbol{B} + \nabla \times \boldsymbol{V})\, dx \tag{4}$$

represent the energy (the incompressibility condition eliminates the thermal energy), the magnetic helicity, and the helicity of the generalized vorticity. We assume periodic boundary conditions for simplicity.

The "Beltrami conditions"

$$\boldsymbol{\omega}_j = a_j \boldsymbol{U}_j \qquad (j = 1, 2)\,, \tag{5}$$

implying the alignment of vortices with the corresponding flows [3,4], yield the simplest stationary solution to (1). For constant a_1 and a_2, (5) translate to [3]

$$(\nabla \times -\lambda_+)(\nabla \times -\lambda_-)\boldsymbol{u} = 0\,, \tag{6}$$

where $\lambda_\pm = [(a_2 - a_1^{-1}) \pm \sqrt{(a_2 + a_1^{-1})^2 - 4}]/2$ and $\boldsymbol{u} = \boldsymbol{B}$ or $\boldsymbol{V}$. The general solution to (6) is the linear combination of two Beltrami fields [5,6] $\boldsymbol{G}_\pm$ obeying

$$\nabla \times \boldsymbol{G}_\pm = \lambda_\pm \boldsymbol{G}_\pm\,. \tag{7}$$

If the system contains harmonic magnetic and flow velocity fields, the eigenvalues $\lambda_\pm$ may be arbitrary real or complex conjugate numbers [5]. For arbitrary constants $C_\pm$, the

magnetic field and the corresponding velocity field may be written as

$$\boldsymbol{B} = C_+\boldsymbol{G}_+ + C_-\boldsymbol{G}_-,$$

$$\boldsymbol{V} = (a_1^{-1} + \lambda_+)C_+\boldsymbol{G}_+ + (a_1^{-1} + \lambda_-)C_-\boldsymbol{G}_-.$$

It appears that the double-Beltrami equilibrium may be "derived" by minimizing the total energy E keeping H_1 and H_2 constant (VP2). Two questions arise: (i) What could be the possible significance of a minimum energy state? (ii) Since E, H_1, and H_2 are on an equal *a priori* footing, all being the exact constants of ideal dynamics, what dictates the choice of using E as the target functional (TF) for minimization? Remember that the target functional E_m for VP1 is not a constant of motion.

If one could define an appropriate "free energy" in a statistical mechanical framework, a minimum energy state (as distinct from a minimum potential energy state) could acquire an important connotation in relaxation theory. Several authors [7] have constructed such a statistical mechanics in a Hilbert space of single-fluid MHD. The success of the procedure is contingent upon finding a suitable free energy function which is bounded from below. When the theory is applied to the current system, we find that the free energy is not bounded in the energy norm because of the ion-flow helicity term (a part of the invariant H_2). Thus the minimum free energy route is not available for relaxed states to emerge in a magnetofluid. This characteristic of the ion-flow helicity will render VP2 to be a mathematically ill-posed problem; a demonstration will be given soon.

To understand the essence of the second question we remind the reader that we are dealing with a near ideal but not a perfect ideal system. The dissipation, however small, is the primary agent for leading the system towards relaxation. Simulations of the single fluid system reveal that during the relaxation process (as the system evolves from an initial state to a relaxed state) the magnetic energy E_m goes through a severe adjustment (decrease) while H_1, the collisionless constant of motion, changes only mildly [8]. This is totally consistent with the demands of the variational principle; the system is driven to equilibrium by changes (even drastic) in the value of TF. Different responses to dissipation have given rise to the notion of "selective dissipation" which may be used to order the "fragility" of the invariants [9]. The most fragile (suffering the most from dissipation) invariant could, then, be used as TF, because it is "least invariant" among invariants. This criterion, however, would pick H_2 as TF because of the higher derivatives in the ion-flow helicity contribution. A variational principle like VP2 with E as TF (and H_1, H_2 as constraints) will be physically unsound and will turn out to be mathematically ill posed.

One cannot get out of this difficulty by simply switching the roles of E and H_2; the latter, though more fragile than the former, is unsuitable as TF because it is not bounded below. In the two fluid dynamics, therefore, the invocation of selective dissipation, by itself, does not lead to a well-posed variational problem; some additional input is sorely needed.

Finding a suitable TF is that required input. It follows from the preceding discussion that the TF cannot be one of the three invariants and it must include higher-order derivatives than any of them. The latter implies that it must be a functional of the coercive form; the meaning of "coercive" will become clear as we proceed.

If the constrained minimization of a functional is to lead us to a relaxed state independent of dissipation (though dissipation is essential for its realization), this functional has to be a measure of dissipation in the controlling dynamics (an almost ideal system is very different from a dissipation dominated system such as a diffusion equation for which the dissipation determines the structure of the relaxed states). Further the TF should also be a measure of turbulence because the relaxed state should be as free of turbulence as possible. For the magnetofluid, these demands almost uniquely lead us to choose the generalized enstrophy (defined as the curl of the canonical momentum)

$$F = \int_\Omega |\nabla \times (\boldsymbol{V} + \boldsymbol{A})|^2\, dx \tag{8}$$

as the desired TF. Note that F is a hybrid functional combining the magnetic and fluid aspects of the plasma and can be equivalently thought of as the energy associated with the generalized magnetic field seen by the ions. This choice may be seen either as an extrapolation to three-dimensional fluids of the two-dimensional fluid dynamics where the fluid enstrophy serves as the TF or equivalently as the magnetofluid reincarnation of the one-fluid Taylor model.

A well-posed variational principle results when we minimize F keeping E, H_1, and H_2 constant. The general solution for arbitrary values of the three invariants is not necessarily an equilibrium solution. We find that stable equilibrium solutions result when the three invariants obey a certain relationship, i.e., on well-defined surfaces in the space spanned by the invariants. The parameter space for the existence of stable solutions is still very large. Physically it implies that starting from arbitrary values, one of the invariants must adjust (in this case H_2, the most fragile one) as F is minimized so that it can find the value dictated by the original values of E and H_1. If H_2, for instance, is initially far away from its desired final value, the system may not find its way into a relaxed state. These ideas become clearer in later parts of our Letter.

Before developing the new variational principle we construct a simple nontrivial model to bring into focus various concepts that we have introduced.

Let us consider two functionals

$$\mathcal{G}(u) = \int_\Omega |\nabla u(\boldsymbol{x})|^2\, dx, \qquad \mathcal{H}(u) = \int_\Omega |u(\boldsymbol{x})|^2\, dx$$

with $u = 0$ on $\partial\Omega$ where Ω is a bounded domain in $\mathbf{R}^N$. First, we seek a minimizer of $\mathcal{G}(u)$ with the constraint $\mathcal{H}(u) = 1$. This is a well-posed problem; the minimizer is found from the variational principle,

$$\delta[\mathcal{G}(u) - \lambda\mathcal{H}(u)] = 0\,, \tag{9}$$

where λ is a Lagrange multiplier. The Euler-Lagrange equation $-\Delta u = \lambda u$ with the above-mentioned boundary condition constitutes an eigenvalue problem. We can easily show that the eigenvalue $\lambda > 0$. Let λ_j be an eigenvalue and φ_j be the corresponding normalized eigenfunction ($\|\varphi_j\|^2 \equiv \int_\Omega |\varphi_j|^2\, dx = 1$). With setting $u = a\varphi_j$, and demanding $\mathcal{H}(u) = 1$, we obtain $a = 1$ and $\mathcal{G}(u) = \lambda_j$. The smallest λ_j, then, yields the minimum $\mathcal{G}(u)$.

The complementary problem of finding a minimizer of $\mathcal{H}(u)$ with the restriction $\mathcal{G}(u) = 1$ turns out to be ill posed. The inherent pathology of the problem is exposed when we set up the variational principle $\delta[\mathcal{H}(u) - \mu\mathcal{G}(u)] = 0$ (μ is a Lagrange multiplier) and analyze the Euler-Lagrange equation $-\Delta u = \mu^{-1}u$. Let $\mu^{-1} = \lambda_j$ (an eigenvalue of $-\Delta$), and $u = a\varphi_j$. The condition $\mathcal{G}(u) = 1$ yields $a = \lambda_j^{-1/2}$, and $\mathcal{H}(u) = 1/\lambda_j$. Hence, the minimum of $\mathcal{H}(u)$ is achieved by the largest eigenvalue that is unbounded, viz., $\inf\mathcal{H}(u) = 0$ and the minimizer $\lim_{\lambda_j\to\infty}\lambda_j^{-1/2}\varphi_j = 0$ is nothing but the minimizer of $\mathcal{H}(u)$ without any restriction. The constraint $\mathcal{G}(u) = 1$ plays no role in this minimization problem [10].

These examples demonstrate that unless the constraints are less fragile (fragility being determined by the number of derivatives) than TF [10], the constrained minimization is meaningless at best. The variational principle simply does not "see" the constraint because an infinitesimal perturbation with a small length scale can contribute any required value to the constraint.

If the second problem is to be properly posed, i.e., for it to lead to a nontrivial manifestation of constrained minimization, the mathematical requirement is the existence of a higher order (coercive) TF. Generalizing our toy model let us affect the minimization of the higher-order functional

$$\mathcal{F}(u) = \int_\Omega |\Delta u|^2\, dx \tag{10}$$

with restrictions $\mathcal{G}(u) = g$ and $\mathcal{H}(u) = h$ and with the additional boundary condition $-\Delta u = 0$ on $\partial\Omega$. Now the well-posed variational principle

$$\delta[\mathcal{F}(u) - \mu_1\mathcal{G}(u) - \mu_2\mathcal{H}(u)] = 0 \tag{11}$$

yields the Euler-Lagrange equation

$$(-\Delta - \lambda_+)(-\Delta - \lambda_-)u = 0$$

with $\lambda_\pm = (\mu_1 \pm \sqrt{\mu_1^2 + 4\mu_2})/2$. The general solution is $u = C_+\varphi_+ + C_-\varphi_-$, where $\varphi_\pm$ are the eigenfunction of $-\Delta$ belonging to the eigenvalue $\lambda_\pm$. The set of double eigenfunction solutions extends the single eigenfunction solution of the previous problems. However, by adjusting either g or h, the general solution can collapse to a single eigenfunction solution. If $g = \lambda_1^2 h$ (λ_1 is the smallest eigenvalue), we obtain $C_+^2 = h$ and $C_- = 0$, and the double eigenfunction solution degenerates into $u = h\varphi_1$. This solution gives the minimum of $\mathcal{F}(u)$ under the given constraints.

The addition of a coercive TF, and the subsequent procedure of adjustment, has accomplished two desirable objectives: (i) the pathology of the earlier ill-posed problem is removed and (ii) the direct influence of the added coercive term on the equation of motion is also removed. The desirability of the latter cannot be overemphasized: without this the theory would have become arbitrary and quite undependable. Notice that the matching condition $\mathcal{G}(u) - \mu\mathcal{H}(u) = 0$, converting the solution to a single eigenfunction of $-\Delta$, renders the larger problem entirely equivalent to the original problem $\delta[\mathcal{G}(u) - \mu\mathcal{H}(u)] = 0$ [because both $\mathcal{G}(u)$ and $\mathcal{H}(u)$ are homogeneous quadratic forms] but with perfect mathematical rigor.

We now transplant this procedure to the derivation of possible relaxed states in a two-fluid plasma. The suggested coercive term F [see (8)] is motivated entirely by physics; it is a natural magnetofluid hybrid that is a simultaneous measure of the levels of turbulence and dissipation. It stands to reason that any macroscopic self-organized relaxed state of a near ideal system must minimize this measure of disorder and decay. The constraint functionals will, obviously, be the dynamical invariants in the ideal limit. We shall, then, seek a minimizer of F making sure (by the appropriate adjustment of constraints) that F does not directly influence the minimizer. If the minimizer turns out to be a stable equilibrium of the ideal system, then we would have found a self-organized state. If such an adjustment cannot be found, our quest for a relaxed state will be aborted; the solutions will be nonequilibrium or unstable and will minimize neither dissipation nor turbulence. Physically, the adjustment can occur through the small dissipation inherent in the dynamics. The most fragile constraint (including the highest-order derivatives) is likely to suffer this adjustment.

The minimization of F keeping E, H_1, and H_2 constant is carried out through the variation

$$\delta(F - \mu_0 E - \mu_1 H_1 - \mu_2 H_2) = 0\,,$$

where μ_0, μ_1, and μ_2 are Lagrange multipliers (constants). This variational principle is well posed because of the coercive term F that forces the functional $F - \mu_0 E - \mu_1 H_1 - \mu_2 H_2$ to be convex having a unique minimizer. Calculating independent variations in $\boldsymbol{A}$ and $\boldsymbol{V}$, and after some manipulation, we obtain

$$\nabla\times\nabla\times\boldsymbol{V} - \mu_0\boldsymbol{V} - \mu_2\nabla\times\boldsymbol{V} = -\nabla\times\boldsymbol{B} + \mu_2\boldsymbol{B}\,, \tag{12}$$

$$\mu_0\boldsymbol{V} = \mu_0\nabla\times\boldsymbol{B} + \mu_1\boldsymbol{B}\,. \tag{13}$$

The general solution will be a "triple Beltrami field"—a linear combination of three Beltrami functions [see (7)]:

$$\boldsymbol{B} = \sum_{j=1}^{3} C_j \boldsymbol{G}_j, \qquad \boldsymbol{V} = \sum_{j=1}^{3} D_j \boldsymbol{G}_j, \tag{14}$$

where (13) demands $D_j = [\lambda_j - (\mu_1/\mu_0)]C_j \equiv d_j C_j$.

Since the general triple Beltrami field (14) may not even be an equilibrium state, we must strive to find a subclass which spans the double Beltrami states—known to be stable equilibria for eigenvalues (λ_1 and λ_2) sufficiently small that the resulting magnetic and flow shear do not drive kink and Kelvin-Helmholtz instabilities. We know from the toy model that this may be achieved by the adjustment of the constraints; the generalized helicity H_2, containing the higher-order derivatives, will be affected the most.

Because the relation between $\boldsymbol{V}$ and $\boldsymbol{B}$ includes λ_j ($j = 1, 2, 3$) and μ_ℓ ($\ell = 0, 1, 2$), the algebra for carrying out the adjustment process is rather complicated [6]. Let us consider a simpler case when the system has no harmonic (external) field. Then, the eigenfunctions $\boldsymbol{G}_j$ are orthogonal [5], and we obtain

$$\begin{pmatrix} E \\ H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} 1 + d_1^2 & 1 + d_2^2 & 1 + d_3^2 \\ \lambda_1^{-1} & \lambda_2^{-1} & \lambda_3^{-1} \\ X_1 & X_2 & X_3 \end{pmatrix} \begin{pmatrix} C_1^2 \\ C_2^2 \\ C_3^2 \end{pmatrix}, \tag{15}$$

where $X_j = (1 + \lambda_j d_j)^2/\lambda_j$. Solving (15) for C_1 and C_2 with $C_3 = 0$, we obtain a relation among E, H_1, and H_2. Denoting the matrix on the right-hand side of (15) by M and writing $L = M^{-1}$, $\mu_1' = -L_{3,2}/L_{3,1}$, and $\mu_2' = -L_{3,3}/L_{3,1}$, the equation for $C_3^2 = 0$ reads as

$$E - \mu_1' H_1 - \mu_2' H_2 = 0,$$

which determines the adjusted value of H_2 for a prescribed E and H_1. This "adjustment" is formally equivalent to solving $\delta(E - \mu_1' H_1 - \mu_2' H_2) = 0$ (all functionals are symmetric bilinear forms), giving the double Beltrami equation (6). We have finally obtained the self-organized relaxed states through a mathematically well-posed variational principle.

We must emphasize that the adjustment condition of the last paragraph is not a variational principle to minimize either E (ill posed, because H_2 is "fragile") or H_1 or H_2 (not lower bounded).

We have derived a well-posed variational principle that reproduces the double Beltrami states characterizing the relaxed states of a two-fluid plasma. The coercive target functional (F) is the measure of turbulence (and simultaneously, of the dissipation), and its minimization is exactly the characterization of self-organization. This is not to be confused as selective dissipation of F; selective dissipation occurs in the fragile quantity (primarily H_2) through the adjusting process. The system can self-organize to a quiescent state, minimizing F, if the constants of motion are appropriately adjusted through a weakly dissipative process. This logical characterization of self-organization is naturally built in our formulation of the variational principle. We have shown that the required adjustment is not the minimization of energy. The conventional model of finding an energy minimizer constitutes an ill-posed problem. This mathematical observation has profound consequences suggesting that the energy is not the principal variable whose minimization characterizes self-organization in vortex dynamics. Our general formalism, in the limit of no flow, provides a new interpretation for Taylor relaxation.

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