Relative Dispersion in Fully Developed Turbulence: The Richardson's Law and Intermittency Corrections

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Relative dispersion in fully developed turbulence is investigated by means of direct numerical simulations. Lagrangian statistics is found to be compatible with Richardson description although small systematic deviations are found. The value of the Richardson constant is estimated as $C_2 \simeq 0.55$, in a close agreement with recent experimental findings [S. Ott and J. Mann, J. Fluid Mech. **422**, 207 (2000)]. By means of exit-time statistics it is shown that the deviations from Richardson's law are a consequence of Eulerian intermittency. The measured Lagrangian scaling exponents require a set of Eulerian structure function exponents ζ_p which are remarkably close to standard ones known for fully developed turbulence.

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The statistics of two-particle dispersion is historically one of the first issues which has been quantitatively addressed in the study of fully developed turbulence. This was done by Richardson, in a pioneering work on the properties of dispersion in the atmosphere in 1926 [1], 15 years before the theoretical development by Kolmogorov and Obukhov [2]. Despite this fact, there are still relatively few experimental studies on turbulent Lagrangian dispersion. This is essentially due to the difficulties to obtain Lagrangian trajectories in fully developed turbulent flow. The first studies were done in geophysical flows (see [2] for a review) in which Lagrangian tracers are more easily followed. Recently, the problem was approached in laboratory experiments [3,4] but the results are still not conclusive. Moreover, most of the numerical studies of relative dispersion rely on kinematic simulations in synthetic flows [5,6]. Direct numerical simulations have been done mostly for two-dimensional turbulence [7,8].

The scope of this Letter is to contribute to the understanding of relative dispersion by means of direct numerical simulations of three-dimensional turbulence. In what follows we show the qualitative validity of the Richardson's description, and discuss its limitations as posed by Lagrangian intermittency, whose properties will be investigated in detail.

Richardson's original description of relative dispersion is based on a diffusion equation for the probability density function (pdf) of pair separation $p(\mathbf{r}, t)$ which in the isotropic case can be written as

$$\frac{\partial p(\mathbf{r}, t)}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 K(r) \frac{\partial p(\mathbf{r}, t)}{\partial r}. \qquad (1)$$

The turbulent eddy diffusivity was empirically established by Richardson to follow the "four-thirds law" $K(r) \propto r^{4/3}$. This law is a direct consequence of the small-scale velocity statistics, as was first recognized by Obukhov [9]. Thus, for r within the inertial range, the dimensional analysis gives

$$K(r) = k_0 \varepsilon^{1/3} r^{4/3}, \qquad (2)$$

where ε is the mean energy dissipation and k_0 a dimensionless constant. We note that (2) relies on the connection between the energy dissipation and the velocity correlation functions in a well-developed turbulence, and that the particle dispersion is, of course, observed also in pseudoturbulent synthetic Gaussian velocity fields [5,6,10] for which the concept of energy dissipation is meaningless. Using (2), the solution of (1) for δ-distribution initial condition has the well known form

$$p(\mathbf{r}, t) = \frac{A}{(k_0 t)^3 \varepsilon} \exp\left(-\frac{9 r^{2/3}}{4 k_0 \varepsilon^{1/3} t}\right), \qquad (3)$$

where $A = 2187/2240\pi^{3/2}$ is a normalizing factor. The most important feature of the Richardson distribution (3) is non-Gaussianity with a very pronounced peak at the origin and fat tails. In the past, alternative distributions have been proposed [11,12]. In particular, Batchelor [11] suggested a Gaussian distribution as a consequence of a diffusivity which depends only on averaged quantities. Because the available data are scarce, there is still no general consensus on the real form of pair separation pdf. Recent experimental works [3,4] are in favor of (3).

As a consequence of (1) and (2), Eq. (3) proposes that the dispersion process is self-similar in time; i.e., the scaling exponents of the moments of the separation

$$R^{2n}(t) \equiv \langle r^{2n}(t) \rangle = C_{2n} \varepsilon^n t^{\alpha_{2n}} \qquad (4)$$

have the values $\alpha_{2n} = 3n/2$, as follows from dimensional analysis. All the dimensionless coefficients C_{2n} are given in terms of k_0 from (3) and a single number such as the Richardson constant C_2 (which can be readily measured) is sufficient to parametrize turbulent dispersion. In what follows we shall discuss to what extent this is the case.

The possibility to describe the dispersion process by means of a diffusion equation is based on the physical assumption that the velocity field is short correlated in time.

Indeed, in the limit of velocity field δ-correlated in time (the so-called Kraichnan model of turbulence) the diffusion equation of the type of Eq. (1) becomes exact [13,14]. In the general case of real turbulence with finite time correlations, a theoretical derivation of the general properties of $p(\mathbf{r}, t)$ is still not available. This is because the Lagrangian relative velocity correlation time grows with time, and thus in absence of decorrelation, central limit theorem cannot be applied [14]. The effects of finite correlation time have been recently discussed in [8,15,16].

There is still a large uncertainty on the value of C_2, ranging from $O(10^{-2})$–$O(10^{-1})$ for kinematic simulations [5,6] to $O(1)$ or more in the case of closure predictions [2]. A recent experimental investigation gives the value $C_2 = 0.5$ [4]. The hypothesis of self-similarity is reasonable with a self-affine Eulerian velocity, such as in the case of two-dimensional inverse cascade turbulence, where the dimensional exponents $\alpha_{2n} = 3n/2$ have indeed been found [8]. A recent analysis of a kinematic model with synthetic velocity field has shown that Lagrangian self-similarity is broken in the presence of Eulerian intermittency. In this case the exponents α_n have been found in agreement with the prediction of a multifractal approach for Lagrangian statistics. In particular, the second moment of relative dispersion is not affected by intermittency, i.e., $\alpha_2 = 3$ [10], essentially because it is proportional to ε^1. We recall that Lagrangian intermittency has been observed also in the case of the so-called strong anomalous diffusion [17]. Although in that case the mechanism leading to intermittency is different (there is no scaling invariant flow), the implication for Lagrangian description is identical; i.e., the process cannot be described by a Fokker-Planck equation like (1).

We now turn to our numerical procedure. The turbulent velocity field is generated by direct integration of Navier-Stokes equation in a periodic box of size $L = 2\pi$. The integration is done on a Cray T3E parallel computer by means of a pseudospectral code at resolution 256^3 with $Re_\lambda \simeq 200$. Energy is injected into the flow by keeping the total energy in each of the two first wave number shells constant [18] and is removed by a second-order hyperviscous dissipation.

Passive tracer trajectories are obtained by integrating $\dot{\mathbf{x}}(t) = \mathbf{u}(\mathbf{x}(t), t)$ with the velocity at particle positions obtained by linear interpolation from the nearest grid points. A single run follows the evolution of separations of about 3×10^5 pairs starting from initial separation $R(0) = L/256$ until they have all reached the integral scale. The reported results are obtained after averaging over 10 independent runs.

In Fig. 1 we plot the second moment of relative dispersion $R^2(t)$. The Richardson t^3 law (4) is clearly observable although systematic deviations are detectable, in particular in the compensated plot. These deviations, observed also in kinematic simulations [10] and in two-dimensional turbulence [8], are due to finite size effects. Consider a
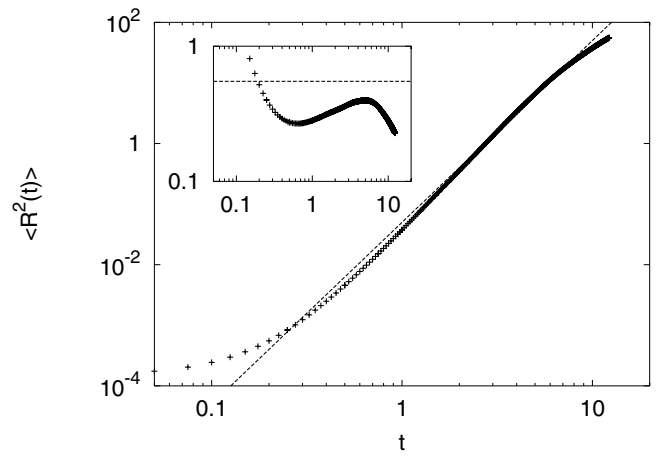

FIG. 1. Relative dispersion $R^2(t)$ versus time t. The dashed line is the Richardson t^3 law. In the inset we show the compensated plot $R^2(t)/\varepsilon t^3$ which should give the Richardson constant C_2. Because of the strong oscillation, a precise estimation of C_2 is difficult.

series of pair dispersion experiments, in each of which a couple of particles is released at time $t = 0$ at initial separation $R(0)$. At a fixed time t one performs an average over all realizations and computes $R^2(t)$. For t small $R^2(t)$ is dominated by the initial distance, so that the $R^2(t)$ curve flattens. For large times some pairs might have reached a separation larger than the integral scale and thus show normal (not Richardson) diffusion, so that the $R^2(t)$ dependence flattens again. Under these conditions, a precise determination of the exponents and coefficients in (4), in particular the Richardson constant C_2, is very difficult.

The distribution of relative separations is plotted in Fig. 2 for three different times. The form of the pdf is very close to the Richardson prediction (3) and excludes
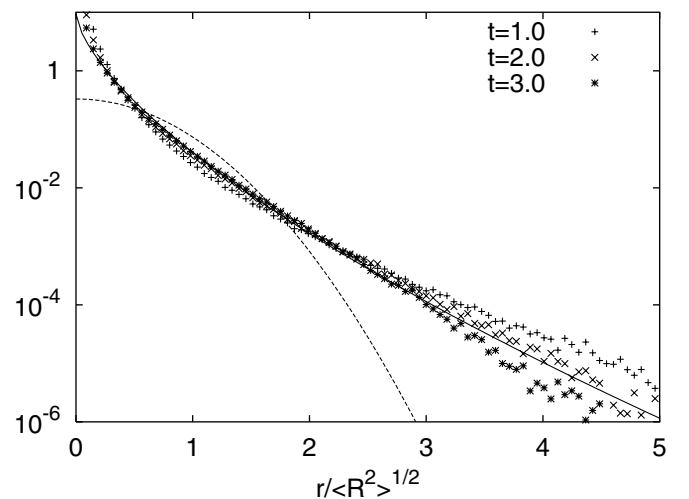

FIG. 2. Probability distribution function of relative separations at three different times. The continuous line is the Richardson prediction (3), and the dashed line is the Gaussian distribution proposed by Batchelor.

other distributions. Our result is the first direct numerical evidence of the substantial validity of Richardson's equation and gives support to recent experimental findings [4]. A closer inspection of Fig. 2 reveals, however, that the self-similar evolution predicted by (1) is not exact. Again, the deviations from the distribution (3) are mostly due to finite Reynolds effects: because of the fat tails, a large fraction of particles exits the inertial range after a very short time.

To overcome these difficulties in Lagrangian statistics, an alternative approach based on exit time statistics has been recently proposed [10,19]. Given a set of thresholds $R_n = \rho^n R(0)$ within the inertial range, one computes the "doubling time" $T_\rho(R_n)$ defined as the time it takes for the particle pair separation to grow from threshold R_n to the next one R_{n+1}. Averages are then performed over many dispersion experiments, i.e., particle pairs. The outstanding advantage of averaging at fixed-scale separation, as opposite to a fixed time, is that it removes crossover effects since all sampled particle pairs belong to the same scales. In the simulations presented here, the value $\rho = 1.2$ is used.

Let us first show how doubling-time analysis can be used for estimating the Richardson constant C_2. Neglecting intermittency, the mean doubling time can be obtained from the first-passage problem for the Richardson diffusion equation (1) as [8]

$$\langle T_\rho(R) \rangle = \frac{\rho^{2/3} - 1}{2k_0 \varepsilon^{1/3} \rho^{2/3}} R^{2/3}. \qquad (5)$$

From (3) and (4) one has $C_2 = \frac{1144}{81} k_0^3$. Comparison with (5) gives

$$C_2 = \frac{143}{81} \frac{(\rho^{2/3} - 1)^3}{\rho^2} \frac{R^2}{\varepsilon \langle T_\rho \rangle^3}. \qquad (6)$$

In the inset of Fig. 3 we plot expression (6) which gives directly the value of C_2. Although the compensation is not perfect, it is possible to estimate the Richardson constant with much better accuracy than from the direct analysis of Fig. 1. The resulting value, $C_2 \simeq 0.55$, even if affected by large uncertainty, is remarkably close to the recent experimental finding [4]. The imperfect compensation is the consequence of intermittency.

Let us now discuss the issue of intermittency in more detail and concentrate on the behavior of the moments of inverse doubling times, $\langle [1/T_\rho(R)]^p \rangle$. We expect for doubling-time statistics a power-law behavior

$$\left\langle \left(\frac{1}{T_\rho(R)} \right)^p \right\rangle \sim R^{\beta_p} \qquad (7)$$

with exponents β_p connected to the exponents α_n in (4) [10]. Negative moments of doubling time are dominated by pairs which separate fast; this corresponds to positive moments of relative separation. Kolmogorov scaling, based on the dimensional analysis, gives $\langle [1/T_\rho(R)]^p \rangle \sim \varepsilon^{p/3} R^{-2p/3}$ so that $\beta_p = -2p/3$. Intermittency can be
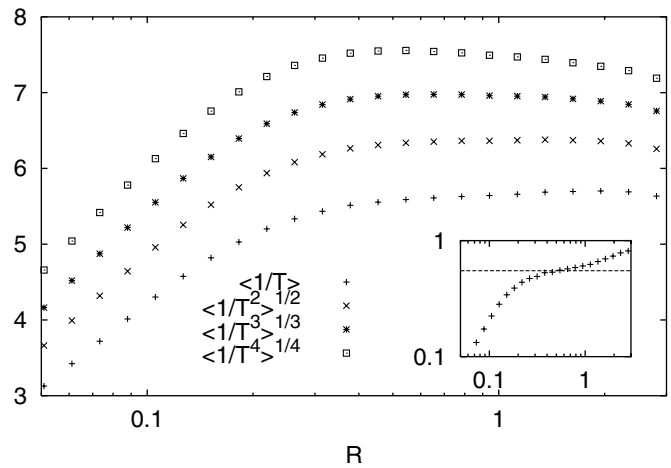

FIG. 3. First moments of the inverse doubling time $\langle [1/T(R)]^p \rangle$ compensated with Kolmogorov scaling $R^{-2p/3}$. Deviations from dimensional compensation are evident, in particular for $p = 4$. In the inset we plot the compensated mean doubling time according to (6) together with the estimate corresponding to $C_2 \simeq 0.55$.

taken into account by using the simple dimensional estimate for the doubling time, $T(R) \sim R/\delta u(R)$, which gives

$$\beta_p = \zeta_p - p, \qquad (8)$$

where ζ_p are the scaling exponents of the longitudinal structure functions. As a consequence of the Kolmogorov "4/5" law, $\zeta_3 = 1$ [20] and the doubling-time exponent not affected by intermittency is $\beta_3 = -2$ [again, the quantity not affected by intermittency in (7) depends on the first power of ε] [21].

In Fig. 3 we plot the first moments of inverse doubling time (7) compensated with the Kolmogorov scaling $R^{-2p/3}$. The quality of the scaling is remarkable, especially if compared with the standard statistics of Fig. 2. This allows the detection of small deviations from dimensional scaling. Indeed, a closer inspection of Fig. 3 reveals that the compensation is not perfect, the deviation being more evident for higher moments; this is an indication for Lagrangian intermittency. A consequence of Lagrangian intermittency is the deviation of exponents in (4) from dimensional prediction and eventually the breakdown of self-similarity. Although this cannot directly be measured within the present numerical simulation, the effect of the intermittency of doubling time on (4) was indeed observed in synthetic simulations [10].

Figure 4 shows some moments of the inverse doubling time, now compensated with best fit exponents β_p. The improvement with respect to Fig. 3 demonstrates that the exponents in (7) are corrected in comparison to dimensional prediction. From the doubling time exponent β_p we can obtain the Eulerian exponents ζ_p by inverting (8). The result is shown in the inset of Fig. 3. For comparison with (8) we also plot the set of Eulerian structure function exponents ζ_p obtained from our velocity field by using extended self-similarity (ESS) technique [22]. The agreement is
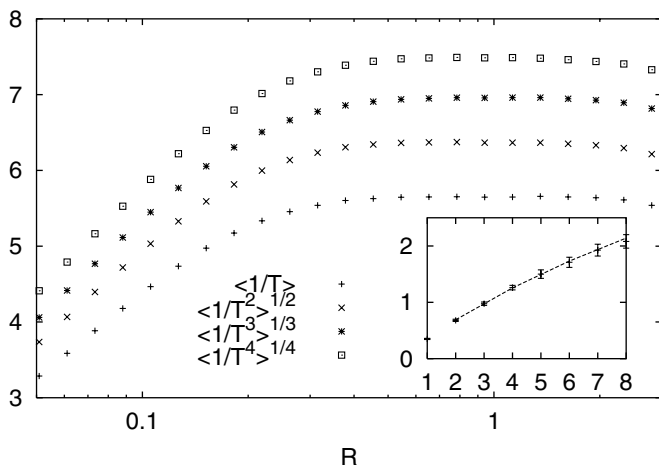
FIG. 4. First moments of the inverse doubling time $\langle[1/T(R)]^p\rangle$ compensated with best fit exponent β_p. Observe the improvement in the compensation with respect to Fig. 3. In the inset we plot the structure function exponents estimated from $\zeta_p = p + \beta_p$. The dashed line represents the Eulerian exponents obtained by ESS analysis.

remarkable even for high-order moments. We stress that, at the present resolution, the scaling of the Eulerian structure function is rather poor, so that only a precise determination of relative exponents (as given by ESS) is possible. The overall position of the ESS curve is given by $\zeta_3 = 1$.

Let us summarize our findings. We have performed direct numerical simulations of a three-dimensional turbulent flow and concentrated on the problem of particles' dispersion. The overall dispersion behavior is well-described by the Richardson's pdf, although some deviations (mostly caused by the finite-Reynolds nature of the simulations) are evident. The use of fixed-scale statistics (doubling-time distribution) instead of fixed-time ones removes to a large extent these restrictions, and gives a possibility to evaluate the Richardson's constant very accurately. Its value is $C_2 \simeq 0.55$, in a close agreement with recent experimental findings. The discussion of the inverse moments of the doubling-time distributions unveils the role of Lagrangian intermittency in the two-particle dispersion. The values of the Lagrangian scaling exponents are connected with the Eulerian structure function exponents ζ_p. The values of ζ_p obtained from the separation statistics are remarkably close to ones obtained by ESS analysis. A natural question is how general is the picture arising from this paper. A further support in favor of generality comes from two-dimensional turbulence in the nonintermittent energy cascade. In that case exit time statistics show no deviations from dimensional predictions. In the future it will probably

be possible to have experimental Lagrangian trajectories in high Reynolds number flows [23]. It would be extremely interesting to check our findings in real fluid turbulence.

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