Consequence of Superfluidity on the Expansion of a Rotating Bose-Einstein Condensate

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We propose an easily detectable signature of superfluidity in rotating, vortex-free gaseous Bose-Einstein condensates. We have studied the time evolution of the expansion of such a condensate after it is released from the confining trap. We find that if such a condensate is not initially rotating, then at some moment it will instantaneously achieve a circular cross section. If the condensate is initially rotating its irrotational flow and the conservation of angular momentum prevent the released condensate from attaining a circular cross section, since the instantaneous moment of inertia is then proportional to the asymmetry of this cross section.

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Recent experiments on trapped atomic Bose gases at very low temperature have revealed the existence of remarkable superfluid effects. Initial evidence of superfluidity came from the observation of quantized vortices [1,2] and of the reduction of dissipative phenomena [3].

Important signatures of superfluidity are also provided by the study of rotating condensates at low angular velocities, in the absence of quantized vortices, where the irrotationality of the flow implies the condition $\nabla \times \mathbf{v} = 0$ everywhere. In particular, the quenching of the moment of inertia provides direct evidence of superfluidity, being related to the suppression of the transverse response of the many-body system [4]. The recent observation of the scissors mode [5] has provided a clear example of this effect, confirming with high accuracy the predictions of theory [6]. The purpose of the present work is to investigate the consequences of superfluidity on the expansion of an initially rotating and vortex-free condensate.

In the absence of rotation the expansion of a condensate after releasing the confining trap is well understood [7,8]. In particular, the shape of a condensate, initially asymmetric in the x-y plane, when released approaches an asymptotic configuration where the role of the long and short axes is inverted with respect to the initial geometry. This inversion occurs because of the larger pressure felt by the system in the short direction where the gradient of the density is higher. As a consequence, the cross section of the condensate in the x-y plane will be instantaneously circular at some intermediate time during the expansion. The time needed to reach this symmetrical configuration depends both on the initial deformation and on the presence of interatomic forces.

If the condensate is initially rotating, new features emerge during the expansion which are the object of the present investigation. In particular, we predict that the condensate will never reach the symmetric configuration, because of the occurrence of a repulsive barrier caused by the irrotationality of the flow. See Fig. 1.

An estimate of the effect is easily obtained by using angular momentum and energy conservation. Angular momentum conservation implies that when the system approaches the symmetric configuration in the x-y plane its angular velocity in the same plane becomes larger and larger as a consequence of the vanishing of the moment of inertia. This is a crucial consequence of the irrotationality of the superfluid motion. On the other hand, the angular velocity cannot increase too much because of energy conservation. By assuming that the initial energy of the system E_0 is mostly converted into rotational energy, one finds $E_0 \sim \Omega_{\rm cr}^2\Theta_{\rm cr}/2$. On the other hand, angular momentum conservation requires $\Omega_0\Theta_0 = \Omega_{\rm cr}\Theta_{\rm cr}$ where Ω_0 and Θ_0 are the angular velocity and the moment of inertia at $t = 0$, while $\Omega_{\rm cr}$ and $\Theta_{\rm cr}$ are the corresponding

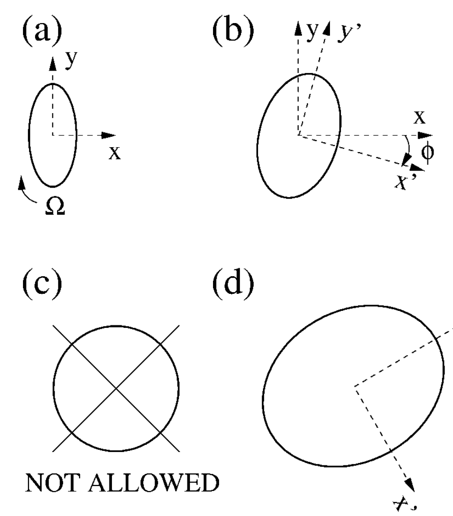

FIG. 1. This figure shows the time evolution of the expanding condensate. (a) The sense of the initial rotation of the condensate and (b) the definition of the rotation angle ϕ are shown above. (c) If the condensate is initially rotating, its cross section can never become circular. (d) The condensate shape at long times.

values at the point of maximum rotational energy (hereafter called critical time). One finds $\Theta_{\rm cr} \sim \Omega_0^2\Theta_0^2/E_0$ and $\Omega_{\rm cr} \sim E_0/(\Omega_0\Theta_0)$. Since in a superfluid the moment of inertia is given by the irrotational value $\Theta = \delta^2\Theta_{\rm rig}$, where

$$\delta = \frac{\langle y^2 - x^2 \rangle}{\langle y^2 + x^2 \rangle} \qquad (1)$$

is the deformation parameter of the cloud calculated with respect to its symmetry axes, and $\Theta_{\rm rig} = Nm\langle x^2 + y^2\rangle$ is the rigid value of the moment of inertia, one concludes that the system will reach a minimum deformation which depends linearly on the initial angular velocity Ω_0, while its angular velocity, at the same time, will behave like $1/\Omega_0$.

The previous discussion suggests the following scenario: the rotating system will first expand in the short direction (we consider here a condensate significantly deformed in the x-y plane at $t = 0$). At a time when the deformation nears its minimum value, the condensate will rotate rapidly causing the inversion of the long and short axes. The condensate then continues to expand along the original long axis. The occurrence of a minimum deformation during the expansion and the corresponding increase of the angular velocity can be regarded as a signature of the superfluid behavior of the system. In fact, a classical rotating gas expands in a very different way, because of the absence of the irrotationality constraint. In the classical case, neglecting the effects of collisions during the expansion, the angular velocity of the sample will decrease smoothly as a function of time according to the law $\Omega_{cl}(t) = \Omega_0/(1 + \Omega_0^2t^2)$.

In the following we will provide a quantitative description of the expansion of a rotating Bose-Einstein condensate at zero temperature in the Thomas-Fermi regime. This regime is relevant to most current experimental situations. The opposite limit, corresponding to the expansion of a rotating ideal gas, will be considered in a separate paper. In the Thomas-Fermi regime the equations of motion take the simplified form of the hydrodynamic equations of superfluids [9]. For the free expansion these equations, valid in the laboratory frame, can be written as an equation of continuity

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0\,, \qquad (2)$$

and a Newton's second-law "force" equation

$$m\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\frac{1}{2}mv^2 + gn\right) = 0\,, \qquad (3)$$

where n and $\mathbf{v}$ are the condensate density and velocity field, respectively, and m is the atomic mass. The factor $g = 4\pi\hbar^2 a_{sc}/m$ in the second equation measures the strength of the atom-atom interaction where a_{sc} is the s-wave scattering length.

The condensate is assumed to be held at equilibrium in a rotating trap just before turning off the external potential. In [10] it was shown that in the rotating frame the stationary solution of the equations of motion still corresponds to a spheroidal form for the density profile. At $t = 0$ we can write

$$n(\mathbf{r},0) = b_0 - a_{x0}x^2 - a_{y0}y^2 - a_{z0}z^2 \qquad (4)$$

and

$$\mathbf{v}(\mathbf{r},0) = \alpha_0\nabla(xy)\,, \qquad (5)$$

where the x-y plane is perpendicular to the axis of rotation. The coefficients b_0, a_{i0}, and α_0 are determined by the number of atoms, N, the trap frequencies, ω_i, and the initial angular velocity, Ω_0, and should fulfill the stationary criteria discussed in [10]. For example, one has the relationship

$$\alpha_0 = \left(\frac{a_{x0} - a_{y0}}{a_{x0} + a_{y0}}\right)\Omega_0\,. \qquad (6)$$

If the angular velocity Ω_0 is small, b_0 and a_{i0} coincide with their values in the absence of rotation and are given by $b_0 = \mu_{\rm TF}/g$ and $a_{i0} = m\omega_i^2/2g$, where $\mu_{\rm TF}$ is the Thomas-Fermi value of the chemical potential. In the same limit of small angular velocity the chemical potential fixes the initial radii, $X_{i0} = (b_0/a_{i0})^{1/2}$ [see Eq. (4)] of the condensate according to the relationships $2\mu_{\rm TF} = m\omega_x^2X_0^2 = m\omega_y^2Y_0^2 = m\omega_z^2Z_0^2$.

The solution of the expanding condensate can be written in the form

$$n(\mathbf{r},t) = b(t) - a_x(t)x^2 - a_y(t)y^2 - a_z(t)z^2 - a(t)xy \qquad (7)$$

and

$$\mathbf{v}(\mathbf{r},t) = \tfrac{1}{2}\nabla[\alpha_x(t)x^2 + \alpha_y(t)y^2 + \alpha_z(t)z^2 + 2\alpha(t)xy]\,. \qquad (8)$$

The condensate occupies the region in space where $n > 0$. Note that the function $b(t)$ defines the density at the center of the trap, $n(0,t) = b(t)$. At $t = 0$ one has $b(0) = b_0, a_i(0) = a_{i0}, \alpha(0) = \alpha_0$, and $a(0) = \alpha_i(0) = 0$.

We assume that the condensate maintains a spheroidal shape as it expands, as shown by Eq. (7). Because of the initial angular velocity in the x-y plane, the symmetry axis will rotate and the density will no longer be diagonal in the x, y coordinates. The angle of rotation can easily be evaluated in terms of the coefficients of (7) by introducing the usual transformation $x = x'\cos\phi + y'\sin\phi$, $y = -x'\sin\phi + y'\cos\phi$, and $z = z'$. This transformation diagonalizes the quantity $s^2 = a_x(t)x^2 + a_y(t)y^2 + a_z(t)z^2 + a(t)xy$ which takes the form

$$s^2 = \tfrac{1}{2}(a_x + a_y)[(x')^2 + (y')^2] + \tfrac{1}{2}[(a_x - a_y)\cos 2\phi - a\sin 2\phi][(x')^2 - (y')^2] + a_z(z')^2 \qquad (9)$$

with the angle of rotation fixed by the relation

$$\tan 2\phi = -\frac{a}{a_x - a_y}\,. \qquad (10)$$

The deformation parameter (1), calculated in the rotating frame, then takes the form (we assume $\delta \geq 0$)

$$\delta = \frac{\sqrt{(a_x - a_y)^2 + a^2}}{(a_x + a_y)}. \qquad (11)$$

The equations of motion for the nine parameters a_i, α_i, a, α, and b are obtained by inserting Eqs. (7) and (8) into the hydrodynamic Eqs. (2) and (3). The equation of continuity yields

$$\dot{b} + (\alpha_x + \alpha_y + \alpha_z)b = 0, \qquad \dot{a}_x + (\alpha_x + \alpha_y + \alpha_z)a_x + 2\alpha_x a_x + \alpha a = 0,$$
$$\dot{a}_y + (\alpha_x + \alpha_y + \alpha_z)a_y + 2\alpha_y a_y + \alpha a = 0, \qquad \dot{a}_z + (\alpha_x + \alpha_y + \alpha_z)a_z + 2\alpha_z a_z = 0, \qquad (12)$$
$$\dot{a} + (\alpha_x + \alpha_y + \alpha_z)a + (\alpha_x + \alpha_y)a + 2\alpha(a_x + a_y) = 0,$$

while the force equation provides four more equations

$$\dot{\alpha}_x + \alpha_x^2 + \alpha^2 - (2g/m)a_x = 0, \qquad \dot{\alpha}_y + \alpha_y^2 + \alpha^2 - (2g/m)a_y = 0,$$
$$2\dot{\alpha} + 2(\alpha_x + \alpha_y)\alpha - (2g/m)a = 0, \qquad \dot{\alpha}_z + \alpha_z^2 - (2g/m)a_z = 0. \qquad (13)$$

Equations (12) and (13) complete the nine first-order ordinary differential equations that we must solve in order to get n and $\mathbf{v}$ as a function of time. An equivalent formalism was developed in [11] to investigate the behavior of nonlinear oscillations carrying angular momentum.

Equations (12) and (13) ensure the conservation of the integrals of motion, i.e., the number of atoms N, the total energy E, and the angular momentum L_z. These quantities can be expressed in terms of our variables. For example, the integral of the density over all space must equal N:

$$N = \frac{8\pi}{15} \frac{b^{5/2}}{(a_x a_y - a^2/4)^{1/2}} a_z^{-1/2}, \qquad (14)$$

while the angular momentum, given by the irrotational law $L_z = Nm\delta^2\Theta_{\rm rig}$, takes the form

$$L_z = \frac{1}{7} mN\Omega\,\delta^2 \frac{b(a_x + a_y)}{a_x a_y - a^2/4}. \qquad (15)$$

The conservation laws (14) and (15) have been used to check the consistency of our numerical solutions.

It is important to note that the equations of motion [Eqs. (12) and (13)] are invariant with respect to the scaling transformation $t \to t/\Lambda$, $\alpha_i \to \Lambda\alpha_i$, $\alpha \to \Lambda\alpha$, $a_i \to \Lambda^2 a_i$, $a \to \Lambda^2 a$, so that the initial size of the condensate can actually be excluded from the solution. For example, the dimensionless quantity δ can be presented as $\delta(\tilde{t})$, where $\tilde{t} = t(2ga_{x0}/m)^{1/2}$. For small initial angular velocities, $\tilde{t}$ coincides with $\omega_x t$ where ω_x is the frequency of the trapping potential. The function $\delta(\tilde{t})$ depends only on the initial shape of the condensate, i.e., on the ratios a_{y0}/a_{x0}, a_{z0}/a_{x0} and on the dimensionless constant $\tilde{\alpha}_0 = \alpha_0(2ga_{x0}/m)^{-1/2}$. The same behavior holds for the angle of rotation $\phi(\tilde{t})$ and for the dimensionless angular velocity $\tilde{\Omega}(\tilde{t}) = d\phi/d\tilde{t} = \Omega(2ga_{x0}/m)^{-1/2}$.

The results of the numerical integration of Eqs. (12) and (13) are presented in Figs. 2–4. We have assumed that initially the condensate has a cigarlike shape elongated along the y direction with $a_{z0}/a_{x0} = 1$, $a_{y0}/a_{x0} = \lambda^2$, and $\lambda = 0.39$. This corresponds to an initial deformation in the x-y plane given by $\delta_0 = (1 - \lambda^2)/(1 + \lambda^2) = 0.74$. Time is measured in units of the dimensionless quantity $\tilde{t}$. Three different values of the parameter $\tilde{\alpha}_0$ have been considered $(0.05, 0.15, 0.45)$, corresponding to different choices of the initial angular velocity and of the trapping parameters. In the presence of rotation the relationships between the deformation parameters of the trap and the ones of the condensate are not trivial.

Using the stationary solutions derived in [10] we find that the above choices for $\tilde{\alpha}_0$ and λ correspond to the

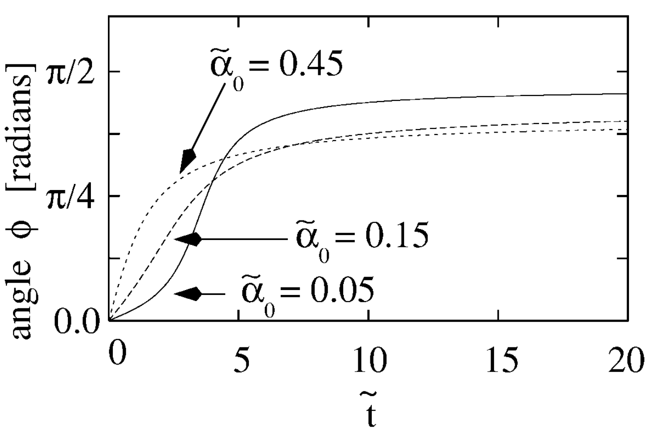

FIG. 2. Angle of rotation of the condensate during the expansion as a function of time for different values of the initial angular velocity. The choice of the parameters is given in the text. Time is given in dimensionless units (see text).

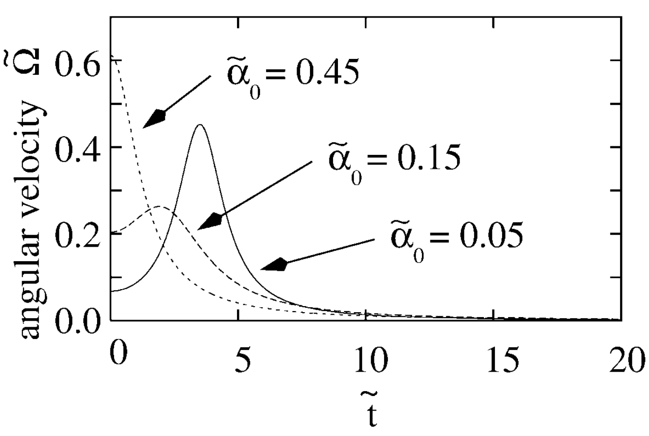

FIG. 3. Angular velocity of the condensate during the expansion as a function of time for different values of the initial angular velocity (see Fig. 2).

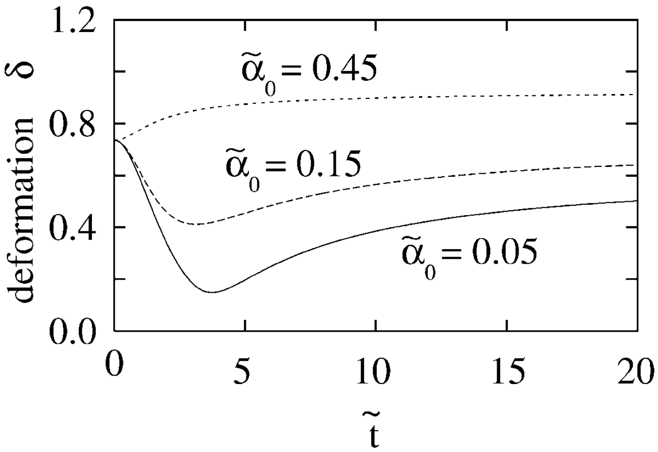

FIG. 4. Deformation parameter of an expanding condensate as a function of time for different values of the initial angular velocity (see Fig. 2).

values $\Omega_0 = 0.07\omega_x, \omega_y = 0.39\omega_x, \omega_z = 1.0\omega_x$; $\Omega_0 = 0.21\omega_x, \omega_y = 0.45\omega_x, \omega_z = 1.0\omega_x$; and $\Omega_0 = 1.2\omega_x$, $\omega_y = 1.41\omega_x, \omega_z = 2.0\omega_x$, respectively. Notice that, except for values of Ω_0 much smaller than the trapping frequencies ω_x, ω_y, the relationship between α_0 and Ω_0 is not linear since the parameters a_{x0}, a_{y0} entering Eq. (6) are renormalized by the rotation of the trap. In particular, the last choice ($\Omega_0 = 1.2\omega_x, \omega_y = 1.41\omega_x$) corresponds to a configuration belonging to the overcritical branch discussed in [10] where the deformation of the condensate has opposite sign with respect to the one of the confining trap.

Figure 2 shows that the angle of rotation $\phi(\tilde{t})$ reaches, for large times, a constant value. This is due to the rapid increase in the condensate size during the expansion which, in turn, causes a fast decrease of the angular velocity. The actual value of the asymptotic angle depends on the initial conditions, different from what happens in the ideal gas where ϕ always approaches the value $\pi/2$. In Fig. 3 we show the angular velocity $\tilde{\Omega}(\tilde{t})$. One can see that, for small values of the initial angular velocity, $\tilde{\Omega}$ exhibits a clear enhancement in the first stage of the expansion. Correspondingly the deformation parameter $\delta(\tilde{t})$ reaches a minimum (see Fig. 4). If the initial angular velocity is large, the minimum is shallow and eventually disappears. In this case also the angular velocity changes its behavior and decreases monotonically with time (see dotted line in Fig. 3). In the recent experiment of Ref. [12], steady rotating states of magnetically trapped condensates were generated using rotating laser beams. The condensates were then imaged after expansion. By using the formalism of Ref. [11], the authors of Ref. [12] have checked that the deformation of the expanding condensate exhibits only minor changes. This result is the consequence of the relatively large values of the initial angular velocity considered in this experiment.

In conclusion, we have shown that the expansion of a rotating condensate reveals dramatic consequences of superfluidity associated with the reduction of the moment of inertia. These effects appear as a sudden increase of the angular velocity accompanied by the occurrence of a minimum in the deformation parameter. These effects, which are particularly pronounced for small initial angular velocities, should easily be observable by taking consecutive images during the expansion of the rotating condensate.

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[1] M. R. Matthews, B. P. Anderson, P. C. Haljan, D. S. Hall, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. **83**, 2498 (1999).
[2] K. W. Madison, F. Chevy, W. Wohlleben, and J. Dalibard, Phys. Rev. Lett. **84**, 806 (2000).
[3] C. Raman, M. Köhl, R. Onofrio, D. S. Durfee, C. E. Kuklewich, Z. Hadzibabic, and W. Ketterle, Phys. Rev. Lett. **83**, 2502 (1999).
[4] G. Baym, in *Mathematical Methods in Solid State and Superfluid Theory,* edited by R. C. Clark and E. H. Derrick (Oliver and Boyd, Edinburgh, 1969); S. Stringari, Phys. Rev. Lett. **76**, 1405 (1996); F. Zambelli and S. Stringari, Phys. Rev. A **63**, 033602 (2001).
[5] O. M. Maragó, S. A. Hopkins, J. Arlt, E. Hodby, G. Hechenblaikner, and C. J. Foot, Phys. Rev. Lett. **84**, 2056 (2000).
[6] D. Gúery-Odelin and S. Stringari, Phys. Rev. Lett. **83**, 4452 (1999).
[7] Y. Castin and R. Dum, Phys. Rev. Lett. **77**, 5315 (1996).
[8] F. Dalfovo, C. Minniti, S. Stringari, and L. Pitaevskii, Phys. Lett. A **227**, 259 (1997).
[9] S. Stringari, Phys. Rev. Lett. **77**, 2360 (1996); F. Dalfovo, S. Giorgini, L. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
[10] A. Recati, F. Zambelli, and S. Stringari, Phys. Rev. Lett. **86**, 377 (2001).
[11] P. Storey and M. Olshani, Phys. Rev. A **62**, 033604 (2000).
[12] K. W. Madison *et al.,* cond-mat/0101051.