Waiting-Time Statistics of Self-Organized-Criticality Systems

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It is argued that a system governed by self-organized-criticality (SOC) dynamics can lack Poisson waiting-time statistics not only when the experimental resolution lies within the self-similar scale range but also if the system is slowly driven in a correlated way. This result thus suggests that waiting time statistics cannot be used as a necessary test for SOC behavior in real physical systems.

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Many complex physical systems seem to exhibit characteristics strikingly similar to those found in systems following the self-organized-criticality (SOC) paradigm. This concept brings together the ideas of self-organization of nonlinear dynamics with the often observed near-critical behavior of many natural phenomena [1]. A large disparity between the time scales associated with the system driving and response is essential for SOC dynamics to exist. This is generally true in systems with instability thresholds: the incoming energy slowly builds up the profiles until the threshold is locally overcome and the system then self-organizes itself via fast relaxation events (avalanches) that dissipate the energy excess across the system. Without need of any tuning, the system reaches a steady state that exhibits the same spatial and temporal self-similar spectra characteristic of critical points in phase transitions as well as spatial and temporal long correlations. Such self-similar spectra and long correlations have indeed been found in a wide variety of physical and biological systems for many years, from earthquakes to turbulent transport in magnetically confined plasmas [2–10]. These signatures are, however, not unique to SOC (even if neccesary) which, together with the lack of a definitive experimental test for SOC behavior in real systems, has raised some criticism about the applicability of this concept in many cases [11].

Recently, a new necessary SOC signature has been proposed, in the context of solar flare dynamics, that seems to have confirmed some of this criticism [12]. It is based on a different type of statistics that deals with the waiting times, i.e., the time intervals between two successive bursts or avalanches. It can be argued that, if their triggerings are not correlated (characteristic built into Ref. [12] many randomly driven sandpile simulations), the process should be somehow related to a Poisson process, and the probability distribution function (PDF) of the waiting times should be an exponential law: $P(\tau) = \bar{\tau}^{-1} \exp(-\tau/\bar{\tau})$. Thus, the argument was made, SOC should be discarded as the underlying dynamics if $P(\tau)$, constructed from experimental data, does not decay exponentially. The existence of extended power laws in the waiting-time PDF of solar flare measurements noticed by several authors [12,13] has therefore been used to argue against the relevance of SOC in the dynamics. It has been also proposed that SOC should be discarded in plasma turbulent transport dynamics in magnetic confinement devices after carrying out the same analysis on edge electrostatic fluctuations from the RFX reversed-field pinch [14]. Such tests must, however, be looked at with extreme care. Some years ago, Christensen *et al.* showed, in the context of a spring-block model for earthquakes [15], that waiting times would follow power-law distributions if only events larger than a certain size are considered. The maximum resolution of the measuring system can therefore account for power laws found in the experimental data. But in this Letter we will show that, even if resolution is so high as to detect events of *all the possible sizes* allowed by the system, nonexponential waiting-time statistics can still be found whenever correlations exist in the external driving. Importantly, the underlying SOC dynamics are not distorted even with this correlated drive.

We have carried out our simulations on a 1D running sandpile [16], consisting of L cells, and with a closed and an open boundary, respectively, located at the first and last cells. The iteration rule goes as follows: at each iteration, U_0 grains of sand are dropped at each cell with probability P_0; whenever the local sand slope, $Z_j = h_j - h_{j-1}$, exceeds some prescribed critical value, Z_c, N_f grains of sand are moved to the next cell. The sandpile reaches the critical state after the incoming sand flux is balanced by the flux leaving the system through the open boundary. To carry out any waiting-time statistics in this sandpile, we first need to decide what the working definition of a waiting time should be in real systems. In the standard Poisson process, the jth event happens at time t_j, but it is considered to be of null temporal duration. The jth waiting time is then computed as $\tau_j = t_j - t_{j-1}$, and the PDF of the random variable τ can then be shown to follow the exponential distribution whenever the right-integrated probability density satisfies:

$$P(\tau > t + s | \tau > s) = P(\tau > t), \quad \forall t, s > 0. \quad (1)$$

In any physical system, the duration of a burst or avalanche is, however, nonzero, being in many cases even larger than the lapses of time between events. Therefore, it is not clear what to consider as a waiting time: some authors use the time interval between triggerings, others the time interval between two consecutive maxima in burst intensity, and finally others consider the lapse of time between the end of a burst and the beginning of the next one (or "quiet time"). We will show that only the quiet time would yield an exponential PDF for noncorrelated triggerings in a SOC system. The reason is that the jth waiting time would be otherwise composed of the lapse of time between the jth and $(j-1)$th avalanches plus either the duration of the jth avalanche or half the durations of both the jth and $(j-1)$th avalanches. However, the PDF of the avalanche durations is known to follow a power law. This will make the PDF of the first two defined waiting times exhibit power-law scaling over the range of values corresponding to durations greater than the quiet times. The three definitions are, however, hard to distinguish if the burst durations are much smaller than the quiet times. Since the average quiet time can be estimated as $\bar{\tau} \sim N_f/LU_0P_0$, it is for instance possible to go from one regime to the other by increasing P_0, as shown in Fig. 1.

Power laws in the quiet-time PDF can also be produced by the effect of the finite resolution of the detection system [15]. To show this, we have computed a series of waiting-time PDFs for several sandpiles with length values extending over 2 orders of magnitude, $L = 100{-}10\,000$, thresholding them in the following way: only those triggering events giving rise to avalanches with durations above a prescribed threshold, d_t, are accounted for. In the runs, we have set $Z_c = 200$, $U_0 = 10$, and $N_f = 30$. The value of P_0 is chosen so that avalanches are triggered at approximately the same rate $\bar{\tau}^{-1}$ in all sandpiles while simultaneously avoiding interferences from avalanche overlapping (this is achieved by setting $P_0 = 5 \cdot 10^{-5}$ for the $L = 400$ run and modifying it for the other cases to keep $\bar{\tau}^{-1}$ constant. The goodness of using the previous estimation, $\bar{\tau}^{-1} \sim LU_0P_0/N_f$ for this task is ascertained after the run is completed by determining $\bar{\tau}^{-1}$ from an exponential fit to the unthresholded quiet-time PDF.) When no thresholding is done, the same clean exponential PDF is obtained in all the cases (see inset of Fig. 2). However, the quiet-time PDF can no longer be fitted by an exponential law, and takes on a power-law behavior, when d_t is sufficiently large as to fall within the self-similar duration range. By "self-similar range" we mean the interval of values over which a power law is obtained in the PDF of the avalanche durations, as it is shown in Fig. 3 (in this way, for the $L = 100$ and $L = 10\,000$ sandpiles, the avalanche duration PDF can be, respectively, fitted by power laws with exponents -3.94 ± 0.12 and -3.82 ± 0.16 over the self-similar ranges beginning at durations $d = 35$ and $d = 800$; an analogous self-similar range is also found in the avalanche size PDFs, with exponents, respectively, now given by -1.53 ± 0.14 and -1.35 ± 0.09 over the size self-similar ranges.). This loss of exponential decay suggests that the triggerings of those avalanches with durations (or sizes, since temporal and spatial scales are intrinsically linked in the SOC state) within the self-similar range are indeed correlated to each other. The triggering of events shorter (or smaller) than those durations (or sizes) in the self-similar range appear, however, to be truly decorrelated of that of longer (or larger) avalanches. Therefore, the calculations suggest that a strong relationship exists between self-similarity and temporal correlation. One way to understand the origin of this relationship might be to realize that, in the critical state, any temporal (or spatial) scale within the self-similar range is strongly correlated with larger and smaller scales: the probability of appearance of an avalanche of some duration (or size) is reduced whenever a longer or shorter (larger or smaller) burst is triggered and,

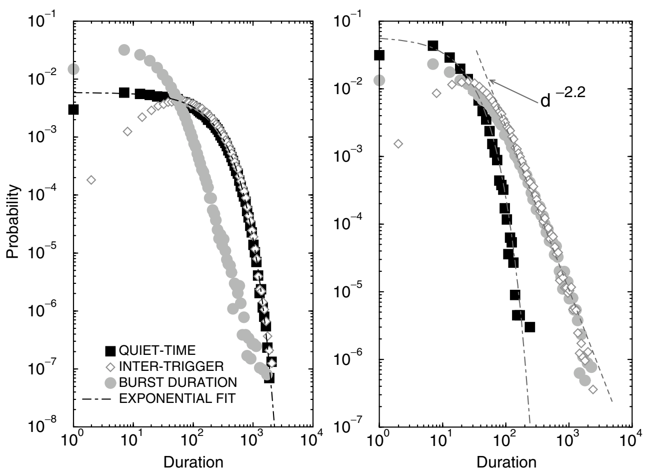

FIG. 1. PDFs for the burst durations, intertrigger times, and quiet times computed for a $L = 400$ sandpile with $Z_c = 200$, $U_0 = 10$, and $N_f = 30$. The left case is run with $P_0 = 5 \times 10^{-5}$ and the right one uses $P_0 = 10^{-4}$.

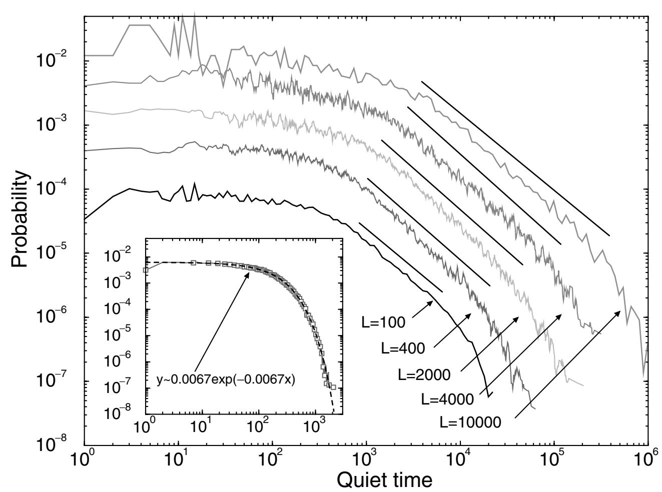

FIG. 2. Quiet times PDFs, for sandpiles with $L = 100$, 400, 2000, 4000, and 10 000, respectively thresholded using $d_t = 30$, 75, 200, 400, and 800. The two upper and two lower PDFs have been, respectively, shifted up and down by a half and a full decade to allow for easier comparison. In the inset, the PDF is shown without thresholding.

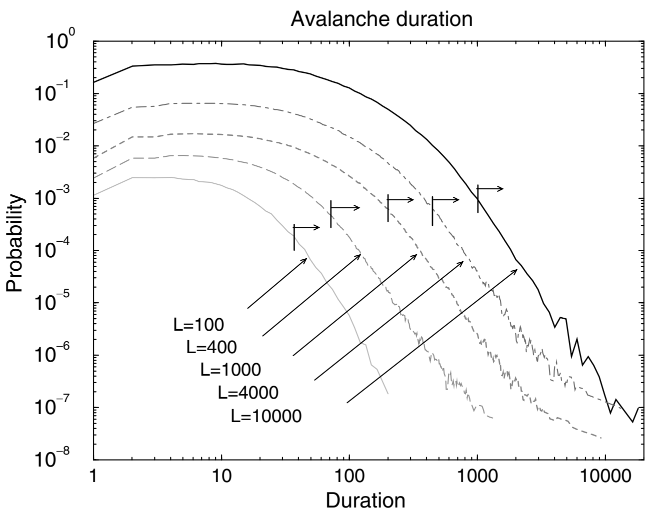

FIG. 3. Duration PDFs for the same cases that Fig. 2. The two upper and two lower PDFs have been, respectively, shifted up and down by a half and a full decade. The beginning of each self-similar range is signaled by an arrow, being, respectively, located approximately at durations 30, 75, 200, 400, and 800.

at the same time, the triggering of an avalanche of that same duration (size) also reduces the probability of appearance of both longer and shorter (larger and smaller) events. However, the self-similar range ceases to exist as the minimum scales allowed in the system (in the case of the sandpile, these minimum scales are determined by those avalanches lasting one iteration and involving just one site) are approached, and the correlations of the shortest (smallest) events with the rest begin to fade away and eventually disappear. And since these shortest (smallest) events are the most frequent ones in the system, the avalanching process will look very similar to a random Poisson process. This is what the numerical experiments suggest, with the correlation between triggerings becoming apparent, in the form of power laws in the quiet-time PDF, only when those scales below the lower cutoff for the power-law scaling in the duration PDF are excluded (interestingly, the existence of this cutoff has also been associated to the same inaccessibility to smaller scales [17]). When avalanches of all durations are, however, included, the more abundant shorter (smaller) events will make the triggering process essentially random, and the exponential decay characteristic of Poisson statistics will dominate.

We can now come to the central question we try to answer in this Letter: namely, if a Poisson quiet-time distribution is a neccesary characteristic of SOC dynamics even if experimental resolution might mask it. From the discussion above, it seems clear that the existence of a minimum meaningful scale in the system (that causes the appearance of a lower cutoff that defines a self-similar range of scales) is responsible for the decorrelation of the smallest events and thus for the exponential decay of the quiet-time PDF. And it also seems clear that any real system must have a smallest meaningful scale: for instance, the size of a single eddy in plasma turbulent transport. But the answer to the fundamental question is still negative, since while the self-organized critical state cannot provide these smallest-scale correlations, the external driving certainly can. In most previous sandpile runs reported the external driving used has been a random noise: for instance, by choosing a random number in [0,1] at each cell and iteration and dropping sand if the number is larger than $(1 - P_0)$. But we have modified this driving by using a time-correlated series for P_0:

$$P_0(t) = P_0(0)[1 - \Delta P_0 \theta_k(t)], \tag{2}$$

where $\theta_k(t) \in [-1, 1]$ is a colored-noise signal with power spectrum decaying as a power law with exponent $k < 0$. By chosing ΔP_0 sufficiently large, a visible power law can be made to appear on the quiet-time PDF without otherwise changing the system dynamics. As an example, we show a $L = 400$ red-driven ($k = -2$) sandpile case with $P_0 = 5 \cdot 10^{-5}$ and $\Delta P_0 = 2$. The correlation time of the noise series is of the order of a few thousand iterations, which seems adequate since $\bar{\tau} \sim 200$ in the randomly driven sandpile. In Fig. 4, the quiet-time PDFs are compared for the red and random cases. The red one exhibits a clear power law that extends for more than a decade while the random case behaves exponentially. The SOC dynamics are, however, unchanged as seen in the inset, where the two PDFs of the avalanche durations are undistinguishable. This can also be seen when comparing the power spectra (or the results from the R/S analysis) of the time series of the total number of overturning cells, which can also be shown to be virtually identical.

As a conclusion, it seems clear to us that the lack of an exponential waiting-time PDF cannot be used to discard SOC dynamics when all other signatures (i.e., power laws in size or duration PDFs, f^{-k} regions in fluctuation power

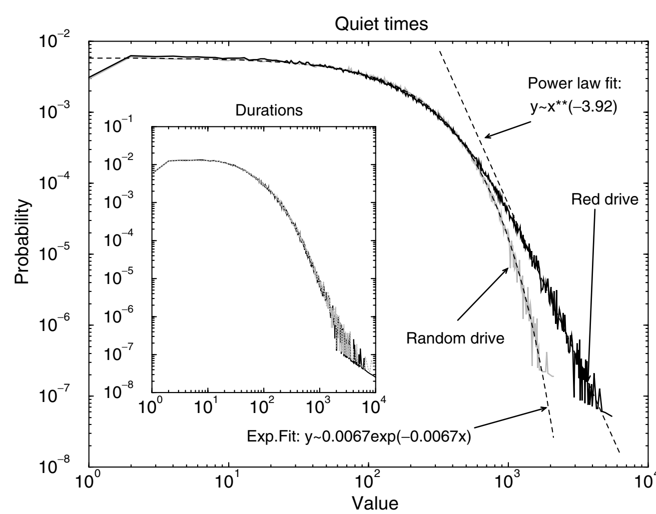

FIG. 4. Quiet times PDF for a $L = 400$ sandpile with random and red drives. Parameters used are $Z_c = 200$, $U_0 = 10$, $N_f = 30$, and $P_0 = 5 \times 10^{-5}$. In the inset, the duration PDFs can be seen to be almost identical.

spectra, or Hurst exponents $H > 0.5$ from R/S analysis) suggest their existence. And this is, in our view, the main result of this paper: an exponential waiting-time PDF is not a necessary characteristic of a SOC system. In cases where avalanche durations are larger than the quiet times, power laws can appear because the waiting-time definition is contaminated by the duration power-law scaling. When the avalanche is much shorter than the quiet times, power laws can still appear if the measurements maximum resolution lies within the self-similar range, since all detected avalanches are then strongly correlated. And even if the experimental resolution is sufficiently high as to detect events of all possible sizes, the lack of exponential decay might simply imply that the system is being driven in a correlated way. The physical origin of the correlated drive, in this last case, will be problem dependent, and it should be identified on a case-by-case basis. For instance, for the solar-flare data records analyzed in Ref. [12], the power laws found in their waiting-time PDF could be reconciled with an SOC model for the flare dynamics, that seems to describe so many other features of the system, if the existence of a correlated drive is conjectured. Identifying the physical origin of such a drive is, however, beyond the scope of the present paper. Although as a first idea, it occurs to us that it might be perhaps related to some random or quasiperiodic process taking place in the Sun or to a coupling to some external perturbation (such as those related to planetary motions, which are a rich source for quasiperiodic disturbances affecting the Sun dynamics).

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