Scaling and Universality in Turbulent Convection

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Anomalous correlation functions of the temperature field in two-dimensional turbulent convection are shown to be universal with respect to the choice of external sources. Moreover, they are equal to the anomalous correlations of the concentration field of a passive tracer advected by the convective flow itself. The statistics of velocity differences is found to be universal, self-similar, and close to Gaussian. These results point to the conclusion that temperature intermittency in two-dimensional turbulent convection may be traced back to the existence of statistically preserved structures, as it is in passive scalar turbulence.

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Heat and momentum transport in slightly heated flows are governed by the Boussinesq equations [1]

$$\begin{aligned} \partial_t T + \boldsymbol{v} \cdot \boldsymbol{\nabla} T &= \kappa \Delta T + f_T\,, \\ \partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} &= -\boldsymbol{\nabla} p - \beta T \boldsymbol{g} + \nu \Delta \boldsymbol{v}\,, \end{aligned} \tag{1}$$

where T is the field of the temperature fluctuations, $\boldsymbol{v}$ is the velocity field, $\boldsymbol{g}$ is the gravitational acceleration, β is the thermal expansion coefficient, and κ, ν are, respectively, the molecular diffusivity and viscosity. The system is kept in a statistically stationary state by the external source of fluctuations f_T. We focus on the statistical properties of temperature excursions at scales larger than the Bolgiano scale l_B—where buoyancy forces balance the inertial ones in the velocity dynamics—yet smaller than the forcing correlation length, L. In that range, temperature fluctuations cascade toward the small scales where they are eventually dissipated by thermal diffusivity. Dimensional arguments based on this phenomenological picture would lead to the Bolgiano-Obukhov scaling, in the range $l_B \ll r \ll L$: $S_n^T(r) = \langle [T(\boldsymbol{r},t) - T(\boldsymbol{0},t)]^n \rangle \sim r^{n/5}$, and $S_n^v(r) = \langle \{[\boldsymbol{v}(\boldsymbol{r},t) - \boldsymbol{v}(\boldsymbol{0},t)] \cdot \hat{\boldsymbol{r}}\}^n \rangle \sim r^{3n/5}$ (see, e.g., Ref. [2] and references therein). Actually, due to the presence of structures of warm-rising or cold-descending fluid—the thermal plumes (see Fig. 1)—the statistics of temperature increments exhibits a nontrivial scale dependence. Indeed, as shown in Fig. 2, moments of temperature increments display a scaling behavior $S_n^T(r) \sim r^{\zeta_n^T}$ characterized by exponents deviating from the dimensional expectations. (Conversely, moments of velocity increments, $S_n^v(r) \sim r^{\zeta_n^v}$, do not show measurable deviations from dimensional scaling, i.e., $\zeta_n^v = 3n/5$.) This *anomalous scaling* is a feature shared by a large class of turbulent systems: understanding the origin of this phenomenon from first principles is a major challenge of turbulence. For passive turbulent transport this problem has been recently solved. Let us briefly recall the main points, referring to Ref. [5]for a comprehensive review. We consider an idealized experiment of turbulent dispersion of a passive tracer, e.g., dye, in a convective flow. The equation that governs the dynamics of the concentration of tracer is

$$\partial_t C + \boldsymbol{v} \cdot \boldsymbol{\nabla} C = \kappa \Delta C + f_C\,, \tag{2}$$

while the velocity field $\boldsymbol{v}$ evolves according to Eq. (1). Although seemingly similar, the dynamics of temperature and concentration fields are radically different: temperature is an *active scalar,* since it affects velocity via the buoyancy forces, whereas concentration is a *passive scalar.* The concentration fluctuations show an anomalous scaling behavior $S_n^C(r) = \langle [C(\boldsymbol{r},t) - C(\boldsymbol{0},t)]^n \rangle \sim r^{\zeta_n^C}$ as well: the exponents ζ_n^C differ from the dimensional expectation. Since $S_n^C(r)$ is a linear combination of various n-point correlation functions of the concentration field $\langle C(\boldsymbol{x}_1,t) \cdots C(\boldsymbol{x}_n,t) \rangle$, the latter has to contain a contribution, denoted as $Z_n^C(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)$, that carries the anomalous scale dependence. In mathematical terms, $Z_n^C(\lambda\boldsymbol{x}_1,\ldots,\lambda\boldsymbol{x}_n) = \lambda^{\zeta_n^C} Z_n^C(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)$. The main point is that the function Z_n^C is characterized by a special dynamical property that distinguishes it from a generic scaling function. Let us remind the reader that the passive scalar equation (2) can be written in the equivalent form $\frac{d}{dt} C = f_C$, where $\frac{d}{dt}$ stands for the total derivative along the particle trajectories defined by the stochastic differential equations $d\boldsymbol{X} = \boldsymbol{v}(\boldsymbol{X},t)dt + \sqrt{2\kappa}\, d\boldsymbol{W}(t)$, where $\boldsymbol{W}(t)$ is Brownian motion. The remarkable result is that $\frac{d}{dt}\langle Z_n^C \rangle_X = 0$, where the total derivative is performed following n particles advected by the flow, and the average is taken over the ensemble of all trajectories. In plain words, Z_n^C is *statistically preserved* by the flow [6–9]. In the specific context of a Gaussian, δ-correlated velocity field—the Kraichnan model—this is equivalent to say that the functions Z_n^C are zero modes of the Fokker-Planck operator for n-particle diffusion [10–12]. An important consequence of statistically preservedstructures is that

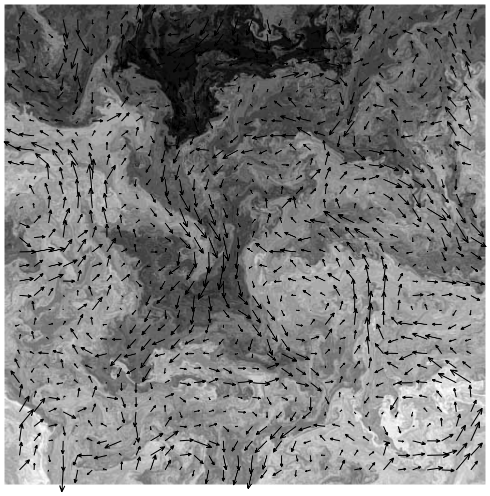

FIG. 1. Snapshot of the temperature and velocity fields. Dark areas identify cold regions. The Boussinesq equations (1) are solved in a two-dimensional doubly periodic domain, with 1024^2 collocation points. The initial conditions were taken from the statistically stationary state of a preliminary run without buoyancy forces and with a random mechanical forcing at small scales (see Ref. [3]). The Bolgiano scale $l_B = \epsilon^{5/4} N^{-3/4} (\beta g)^{-3/2}$ (where ϵ and N are the small-scale kinetic and thermal dissipation rates) coincides with the energy dissipation scale $\eta \approx \nu/\delta_\eta v$ and is approximately equal to the smallest resolved length scale. The Rayleigh number is $\mathrm{Ra} = \beta g L^3 \Delta T \nu^{-2} \approx (L/l_B)^{16/5} \approx 10^7$. Simulations at lower $\mathrm{Ra} \approx 10^6$ show similar results at scales $L \gg r \gg l_B$. The Prandtl number is $\nu/\kappa \approx 1$. Since in two dimensions there is a net energy flux toward the large scales, a statistically steady state requires the momentum equation in (1) to be supplemented by a friction term $-\alpha \boldsymbol{v}$ that drags energy from the gravest modes. As customary, diffusive and viscous terms are replaced by hyperdiffusive $(-\kappa \Delta^2)$ and hyperviscous $(-\nu \Delta^4)$ ones, to confine dissipative effects to the smallest scales.

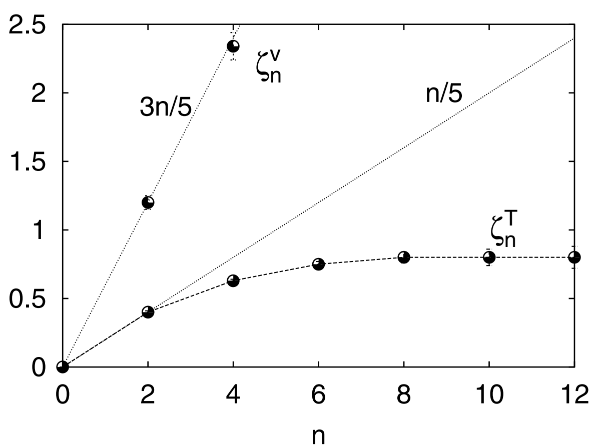

FIG. 2. Scaling exponents of temperature, ζ_n^T, and velocity, ζ_n^v. The straight lines are the dimensional predictions, $n/5$ for temperature, $3n/5$ for velocity. Notice that at orders larger than $n = 8$ the temperature exponents saturate to a constant value $\zeta_\infty^T \simeq 0.8$ [4]. The error bars are estimated by the rms fluctuations of the logarithmic slope. To ensure the statistical convergence of high-order moments we collected 300 snapshots of the fields, spaced by half of the large-eddy turnover time $L/\langle \boldsymbol{v}^2 \rangle^{1/2}$.

the passive scalar scaling exponents are *universal* with respect to the choice of the injection f_C, since the latter does not enter the definition of Z_n^C.

We now turn our attention back to the temperature field. What we have learned in the passive scalar case suggests we should investigate the effect of the external forcing f_T on the scaling exponents. In Fig. 3 we show that the scaling exponents of temperature fluctuations are the same for two different choices of injection terms f_T. Therefore, we conclude that the exponents ζ_n^T are *universal* properties of two-dimensional Boussinesq convection.

The universality of scaling exponents suggests the possibility that a mechanism similar to that at work in passive scalar turbulence might be present in turbulent convection as well. To further pursue this line of thought, we notice that in the case of passive scalars, it is the whole function Z_n^C that is universal with respect to forcing, not only its scaling exponent. It is thus of interest to look at the anomalous part of the temperature correlation function $\langle T(\boldsymbol{x}_1, t) \cdots T(\boldsymbol{x}_n, t) \rangle$ to check whether it is universal. This measurement is unfortunately quite difficult for two reasons. First, the correlation function depends on $2n$ independent coordinates; even if we exploit the statistical symmetries of this function—translational and scaling invariance—and we limit ourselves to the isotropic

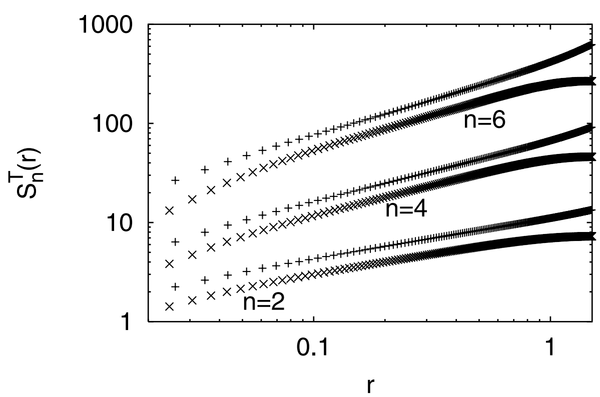

FIG. 3. The moments of temperature differences, $S_n^T(r)$, for $n = 2, 4, 6$, as a function of the separation r. Note the parallelism between curves of the same order n, within the scaling range. The two sets of curves are generated by two different kinds of injection mechanisms. In the first case ($\times$), f_T is a random Gaussian forcing, with correlation $\langle f_T(\boldsymbol{r}, t) f_T(\boldsymbol{r}', t') \rangle = F_T(|\boldsymbol{r} - \boldsymbol{r}'|)\delta(t - t')$, where F_T decays with the characteristic scale L (approximately one-fourth of the box size); in the second case ($+$), the system is driven by the term $f_T = \gamma \boldsymbol{g} \cdot \boldsymbol{u}$, which mimics the effect of a mean temperature gradient on the transport of temperature fluctuations. We consider only the isotropic contribution to the statistics, by averaging over all directions of the separation $\boldsymbol{r}$. For orders equal or larger than $n = 8$ all exponents collapse—within error bars—on the saturation value $\zeta_\infty^T \simeq 0.8$. The curves have been multiplied by appropriate numerical factors for viewing purposes. The equality of the scaling exponents ζ_n^T for the two types of forcing has been checked by computing the logarithmic slope $d \ln S_n^T(r)/\ln r$ (not shown).

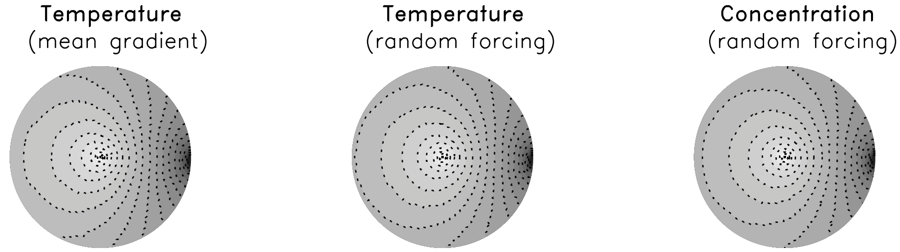

FIG. 4. The functions $\sigma_T(r/R, \theta) = S^T_{2,2}(\boldsymbol{R}, \boldsymbol{r})/S^T_4(R)$ (left and center) and $\sigma_C(r/R, \theta) = S^C_{2,2}(\boldsymbol{R}, \boldsymbol{r})/S^C_4(R)$ (right), in polar coordinates $0 < r/R < 1$ and $0 < \theta < 2\pi$, where $\theta = \cos^{-1}(\hat{\boldsymbol{R}} \cdot \hat{\boldsymbol{r}})$. The color is white where the function is zero, black where it is equal to unity. The function has a minimum at the origin, $\sigma(0, \theta) = 0$, and a maximum $\sigma(1, 0) = 1$ at $\boldsymbol{r} = \boldsymbol{R}$.

contribution—by averaging over all configurations differing only by a rigid rotation—there will still be $2n - 4$ degrees of freedom; in the most favorable case, $n = 4$ (for $n = 2$ deviations from dimensional scaling are not detectable), the configuration space has four dimensions, which makes it quite untractable. Second, the anomalous part of the correlation function is hidden among several other contributions: it can be extracted only by taking proper linear combinations, as, for example, in the case of $S^T_n(r)$. To circumvent, at least partially, those problems, we focus on a particular observable, $S^T_{2,2}(\boldsymbol{R}, \boldsymbol{r}) = \langle [T(\boldsymbol{R}, t) - T(\boldsymbol{0}, t)]^2 [T(\boldsymbol{r}, t) - T(\boldsymbol{0}, t)]^2 \rangle$, which is still anomalous, yet it has a nontrivial geometrical content. Since for $\boldsymbol{r} = \boldsymbol{R}$ it reduces to the usual $S_4(R)$ we can write its functional dependence as $S^T_{2,2}(\boldsymbol{R}, \boldsymbol{r}) = S^T_4(R)\sigma_T(r/R, \theta)$, where θ is the angle between the directions of $\boldsymbol{r}$ and $\boldsymbol{R}$. Since the scaling exponent of $S^T_4(R)$ is universal, the bottom line is whether the "angular" part $\sigma_T(r/R, \theta)$ is universal as well. In Fig. 4 we show a plot of the function σ for the two different injection mechanisms (left and center). The similarity between the two pictures points to the conclusion that the anomalous part of the correlation function is again universal.

This result leads us to conjecture that statistically preserved structures $Z^T_n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ might exist also for temperature: it is natural to define them by the property $\frac{d}{dt}\langle Z^T_n \rangle_X = 0$, as in the passive scalar case. Notice, however, that, since temperature is an active scalar, this definition does not automatically ensure the universality of Z^T_n. Indeed, even if the forcing does not appear explicitly in the definition of Z^T_n, the statistics of the trajectories $\boldsymbol{X}(t)$ in principle depends on f_T, via the action of T on $\boldsymbol{v}$. Therefore, should we accept the existence of statistically preserved structures, the universality of anomalous temperature correlations requires us to postulate that the whole statistics of $\boldsymbol{v}$ is universal as well. In Fig. 5 we show that this is indeed the case.

Statistically preserved structures for temperature fluctuations entail another interesting consequence: since Z^T_n is defined entirely in terms of the (universal) statistics of particle trajectories, and those are the same for both temperature and concentration, we expect that $Z^T_n = Z^C_n$. In Fig. 6 we show that the scaling exponents of temperature and concentration are equal, as expected. As for the "angular part" of Z^T_n, it is quite similar to that of Z^C_n (see Fig. 4, center and right). This is further indirect evidence for the existence of statistically preserved structures for temperature statistics.

In conclusion, the global picture of scaling and universality in two-dimensional turbulent convection is as follows.

Velocity statistics is strongly universal with respect to the external driving: probability density functions of velocity fluctuations are self-similar, and close to a Gaussian distribution, independently of the choice of f_T. This is most likely a consequence of the observed universal Gaussian behavior of the inverse energy cascade in two-dimensional Navier-Stokes turbulence [13,14]. Indeed, velocity fluctuations in two-dimensional convection also arise from an inverse energy cascade which is driven now by buoyancy forces. At variance with the usual

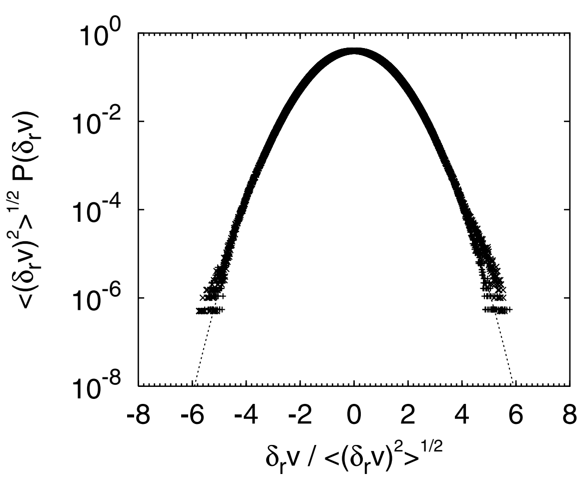

FIG. 5. Probability density function of longitudinal velocity increments $\delta_r v = [\boldsymbol{v}(\boldsymbol{r}, t) - \boldsymbol{v}(\boldsymbol{r}, 0)] \cdot \hat{\boldsymbol{r}}$, rescaled to their standard deviation $\langle (\delta_r v)^2 \rangle^{1/2}$. We show two sets of data obtained by driving the system with random Gaussian forcing ($\times$), and by $f_T = \gamma \boldsymbol{g} \cdot \boldsymbol{u}$ (+). Here $r = 0.2$, inside the scaling range. At different r the rescaled probability density functions collapse onto each other, as expected for a self-similar statistics. The Gaussian density function is shown as a dotted line, for comparison.

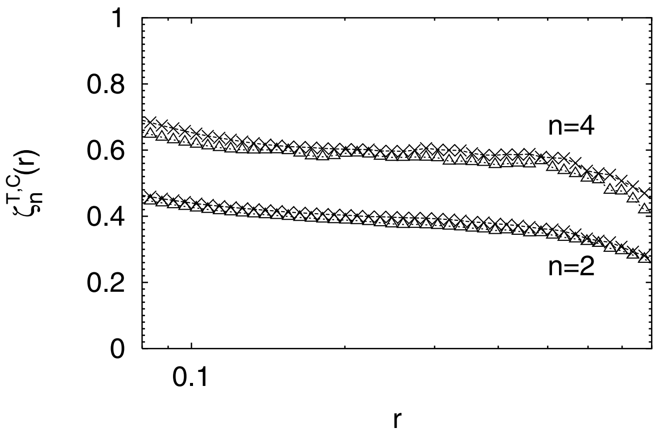

FIG. 6. Local scaling exponents of temperature ($\times$) and concentration ($\triangle$) fluctuations, $\zeta_n^{T,C}(r) = d \ln S_n^{T,C}(r)/d \ln r$. Temperature and concentration are driven by independent Gaussian random forcings.

Navier-Stokes inverse cascade, the energy injection now is not restricted to small scales. Indeed, the energy input rate $\varepsilon(r) = \beta \boldsymbol{g} \cdot \langle \boldsymbol{v}(\boldsymbol{r}, t) T(\boldsymbol{0}, t) \rangle$ grows with the scale as $\varepsilon(r) \sim r^{4/5}$. This scale-dependent input induces the observed scaling $S_n^v(r) \sim [\varepsilon(r) r]^{n/3} \sim r^{3n/5}$. Temperature statistics shows anomalous scaling. This stems from the existence of statistically preserved structures, whose existence explains the observed universality of anomalous temperature correlation functions and the equality between temperature and concentration anomalies.

Let us point out that the equivalence of the statistics of an active scalar, as temperature, to that of a passive scalar, as concentration, depends crucially on the universality of the whole velocity statistics found here. That could, however, be a nongeneric phenomenon in active scalar turbulence and depend on the specific form of the feedback of the scalar field on the velocity. For example, in three-dimensional turbulent convection the Navier-Stokes equations are characterized by a direct and intermittent energy cascade, the whole velocity statistics might then be nonuniversal, and a new type of universality might emerge.

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