Large Scale Structures, Symmetry, and Universality in Sandpiles

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(Received 21 May 2001; published 22 January 2002)

We introduce a sandpile model where, at each unstable site, all grains are transferred randomly to downstream neighbors. The model is local and conservative, but not Abelian. This does not appear to change the universality class for the avalanches in the self-organized critical state. It does, however, introduce long-range spatial correlations within the metastable states. For the transverse direction $d_\perp > 0$, we find a fractal network of occupied sites, whose density vanishes as a power law with distance into the sandpile.

DOI: 10.1103/PhysRevLett.88.054302 PACS numbers: 45.70.Ht, 05.65.+b, 89.75.Da

One of the puzzling questions about macroscopic complex phenomena concerns the mechanisms responsible for the large spatially correlated structures that are often seen in nature. It has been proposed that self-organized criticality (SOC) [1] may be one mechanism in which the intermittent, scale-free threshold dynamics of a slowly driven system is intimately linked to the emergence of long-range spatial (and temporal) correlations in it [2]. An obvious candidate for this picture would be the stick-slip dynamics of earthquakes, described by the Gutenberg-Richter power law distribution for seismic moments, and faults, which form a fractal pattern in the crust of the earth. However, the simple sandpile, or earthquake models do not clearly show large scale structures. Furthermore, although many macroscopic systems show bursty transport phenomenology, a general feature of SOC, the link is not yet established because questions of robustness and universality are not yet resolved. Since the number of possible models that may be studied numerically is inexhaustible, it is essential to determine the symmetry (or other) criteria for universality [3] and robustness of SOC.

Here we propose what may be the simplest sandpile model that gives large spatially correlated structures. For transverse dimension $d_\perp > 0$ the avalanches in the model have a scale-free distribution with critical coefficients that are determined numerically to be in the same universality class as the Abelian stochastic directed sandpile model (A-SDM) [4–6]. There it has been proven that no spatial correlations exist in the steady-state metastable configurations [5]. The model we introduce is closely related to the A-SDM, but a change in the rule for updating unstable sites breaks the Abelian symmetry. The Abelian symmetry refers to the fact that the order for updating the unstable sites has no effect on the final state that is reached.

This symmetry breaking introduces obvious large scale structures, consisting of a fractal network of occupied sites, within the metastable states that are reached in the steady state, as shown in Fig. 2. These spatial correlations are not present when the symmetry is restored. The avalanches change the fractal network configuration slowly, just as earthquakes change the configuration of faults slowly. During a single or a few events it might appear falsely that the network configuration is static or "preexisting."

Since breaking the Abelian symmetry has no effect on the critical exponents for the avalanches, for $d_\perp > 0$, there is universality and robustness for the dynamics. However, two systems in the same universality class with respect to the scaling behavior of avalanches show totally different structures of the metastable states, with one being completely uncorrelated and the other having merging channels of occupied sites, or fractal networks at large scales, with a power law decay of density.

Consider a two dimensional square lattice as shown in Fig. 1. The direction of propagation is labeled by $x_\parallel$, with $0 \leq x_\parallel < L$. The transverse direction is labeled by $x_\perp$, with periodic boundary conditions. On each site, an integer variable $z(x)$ is assigned. The ith grain is added to a randomly chosen site x_i on the top row, so $x_{i\parallel} = 0$. There $z(x_i) \rightarrow z(x_i) + 1$. When any site acquires a height greater than z_c it topples, transferring all the grains at that site, i.e., $z(x) \rightarrow 0$ for $z(x) > z_c$. Each grain from a toppling site is given equal probability to go to either downstream nearest neighbor, independent of where the other grains from the toppling site are placed. For each toppling

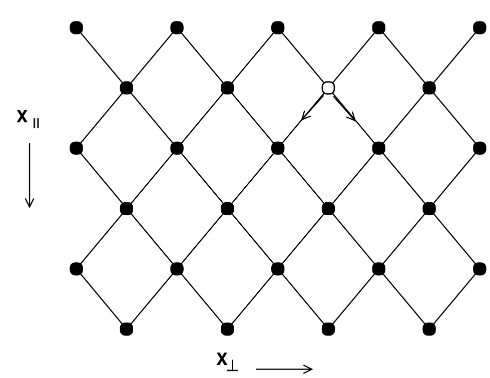

FIG. 1. The model in $d_\perp = 1$. All grains from unstable sites in row $x_\parallel$ are thrown randomly onto neighboring sites in the next row $x_\parallel + 1$.

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event, the total number of grains is conserved. This is true except at the open boundary $x_{\|} = L - 1$ where toppling sites simply discharge their grains out of the system.

Sites are relaxed according to a parallel update until there are no more unstable sites, and the properties of the resulting avalanche are recorded. Then a new avalanche is initiated by adding a single grain to a randomly chosen site on the top row, $x_{\|} = 0$. An avalanche can be characterized by, e.g., its distance, the largest $x_{\|}$ row affected, its size, s, the total number of grains thrown in toppling events, the number of update steps, t, and the maximum number of grains thrown at a single site which topples, $n_{\max}$. The fact that z is set to zero at a toppling site makes the model non-Abelian. The corresponding rule for the A-SDM for an unstable site is, e.g., $z(x) \rightarrow z(x) - 2$.

In a recent work, Dhar [7] has shown that the stochastic Manna model [8], where a fixed number of grains are removed from toppling sites, exhibits the Abelian property and is a special case of the Abelian distributed processors model. This property was used to solve analytically for the critical state properties of the A-SDM, since in that case the Abelian property makes the appropriately mapped dynamics invertible [5], which leads to the product measure property of the metastable states, and the solvability of the A-SDM. However, when all grains from unstable sites are removed in toppling then the model is not Abelian anymore, and these specific analytical methods do not apply.

It is straightforward to generalize the definition of our non-Abelian sandpile model to higher dimensions, with the number of directions transverse to the direction of propagation being $d_{\perp}$. The threshold z_c can be chosen either to scale with dimension as $z_c = 2d_{\perp} + 1$ or it can remain constant at $z_c = 1$. Within numerical error, the same scaling exponents and similar spatial structures are observed under both conditions [9]. For simplicity, the results presented here in the figures refer to the model with $z_c = 1$ in all dimensions.

We performed numerical simulations on systems ranging from $L = 1024$ to $L = 32\,768$ for $d_{\perp} = 0$, $L = 1024$ to $L = 8192$ for $d_{\perp} = 1$, and $L = 256$ to $L = 2048$ for $d_{\perp} = 2$. Statistics were collected in the steady state for at least 10^7 avalanches for each system size. The transverse lengths were chosen large enough so that the avalanches never wrapped around on themselves.

First, we discuss the case $d_{\perp} = 1$. From the dynamical rules it is clear that the avalanches must, themselves, be essentially compact. Thus each avalanche sweeps out areas of the lattice leaving empty sites. At the edges of the avalanche, sites occupied with grains may remain. Thus in the stationary state the structure of the sandpile will consist roughly of empty areas bounded by wandering paths of occupied sites, which can branch and recombine. At $x_{\|} = 0$, where the grains are added, the network of grains is dense, but pushing into the sandpile it becomes coarser and coarser. This coarsening reflects the fact that the rarer avalanches that reach farther into the system are bigger and wider and thus leave traces at their edges that are farther apart. A steady-state sandpile configuration is shown in Fig. 2.

The average density of sites occupied with grains scales with distance from the edge, $x_{\|} = 0$, where grains are added, as $\rho(x_{\|}) \sim x_{\|}^{-\alpha}$, with $\alpha = 0.45 \pm 0.02$. This power law behavior comes about because the various channels merge in a self-similar manner, due to the power law distribution of distances, $x_{\|}$, that avalanches reach into the system, as discussed later. Making a vertical cut in the system from the top, $x_{\|} = 0$, to $x_{\|} = l$, the average number of occupied sites which intersect the cut scales as $N_c \sim l^{d_c}$, with $d_c = 1 - \alpha$. Since $d_c < 1$, due to the merging of channels, the networks are fractal. This fractal network of grains is essential for maintaining the steady state of SOC, providing a drainage outlet for grains to be transported from the top to the bottom of the system. The situation for the A-SDM is completely different. In that case, the density of occupied sites is $1/2$ for $d_{\perp} = 1$, and the occupation numbers for sites are completely uncorrelated, being described by a product measure in the steady state [5].

Despite these vast differences in steady-state configurations between the Abelian model and our model, the distribution of avalanche sizes and distances in the steady state of SOC exhibits finite size scaling (FSS) with the same critical exponents, within numerical accuracy. The FSS ansatz for the probability distribution of avalanche sizes, s, in a system of size L can be written as

$$P_L(s) \sim s^{-\tau_s} G\left(\frac{s}{L^D}\right), \tag{1}$$

where G is some scaling function and τ_s and D are critical exponents describing the scaling of the distribution function. The probability distribution of avalanche durations,

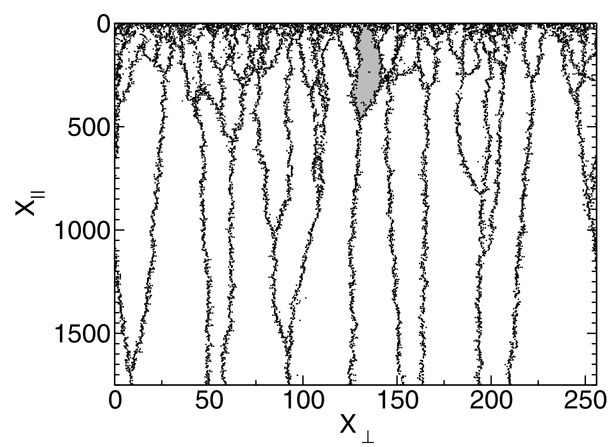

FIG. 2. A steady-state configuration of our $d_{\perp} = 1$ sandpile model showing sites occupied with grains forming a network that can transport grains from one end of the sandpile to the other. Note that the network becomes coarser going into the system as the various channels merge. The shaded area indicates the sites which toppled in the preceding avalanche.

$P_L(t)$, is described by a similar equation with exponents τ_t and D_t. A similar power law distribution also describes the distribution of avalanche distances, $P_L(x_\parallel)$.

All the critical exponents characterizing avalanches in the A-SDM have been determined analytically [5,6] and numerically [4,6,10]. We find numerically for our non-Abelian model in $d_\perp = 1$ that $\tau_s = 1.43 \pm 0.02$, $\tau_t = D = 1.75 \pm 0.01$, compared to the analytic values for the A-SDM $\tau_s = \frac{10}{7}$ and $\tau_t = D = \frac{7}{4}$. This is demonstrated in Fig. 3 where numerical results for the size distributions of avalanches, for both the A-SDM and our non-Abelian model, are presented.

Configurations in the steady state of our sandpile model in $d_\perp = 2$ exhibit the type of domain tube structure with walls separating the different domains. The tube domains get wider as they go into the system, indicating a vanishing density of occupied sites and a fractal network structure. Again, a numerical analysis of the avalanche size distribution using finite size scaling gives critical exponents $\tau_s = 1.5 \pm 0.03$ and $D = 2.0 \pm 0.02$, the same values as determined analytically, e.g., $\tau_s = \frac{3}{2}$ and $D = 2$, for the A-SDM [5,6]. Similarly for the distribution of avalanche durations, t, we observe finite size scaling with exponents $\tau_t = 2.0 \pm 0.02$, and $D_t = 1$. There is no such tube structure, though, in the A-SDM.

The situation in $d_\perp = 0$ is more subtle and less robust. This model is a chain of sites. Since each site has only one downstream nearest neighbor, the dynamical rules of the model must be specified in a slightly different way. We allow grains to be distributed to both the nearest and next-nearest neighbors down the chain and consider two ways in which the relaxation of critical sites can be ordered. With a parallel update rule all critical sites are relaxed simultaneously. In this case, the time in terms of the parallel update at which a site can become critical is not equal to its row number and sites may topple many times during an avalanche, e.g., at update time $t = 2$ from the beginning of the avalanche, the avalanche site $x_\parallel = 3$ may relax.

We can also define another distinct model by updating only the site $x_\parallel = t$, so at each time step only one site is updated. Multiple toppings cannot occur in that case. These two models lead to different sets of critical exponents for the avalanches in $d_\perp = 0$.

The parallel update dynamics yield the same avalanche exponents as the Abelian model that was studied by Kloster *et al.* [6] (see Fig. 4). We found good data collapse for system sizes ranging from $L = 1024$ to $L = 32\,768$, for both the size, s, and the duration (number of parallel update steps), t, of the avalanches. However, the average occupancy of sites in the steady state does not decay to zero as $x_\parallel$ increases, as in our model in higher dimensions. Instead it is constant $\rho(x_\parallel) = \frac{1}{4}$, apart from close to $x_\parallel = 0$ where the average occupancy adjusts exponentially from the initial value of $\frac{1}{2}$. Occupied sites do not appear to be spatially correlated. Thus the behavior of the model is very similar to its Abelian relative [6] and appears to be in the same universality class with exponents $\tau_s = 1.33 \pm 0.02$ and $\tau_t = D = 1.5 \pm 0.01$.

In the parallel update dynamics, as more than one site can topple at each time step and the location of the topplings is allowed to vary, the avalanches themselves have internal structure. On average the active front moves through the system with velocity 1.5 (i.e., advances three lattice spaces in two parallel updates, on average). Around this average the active sites are split into a series of smaller fronts which spread, branch, and recombine (see Fig. 4 inset).

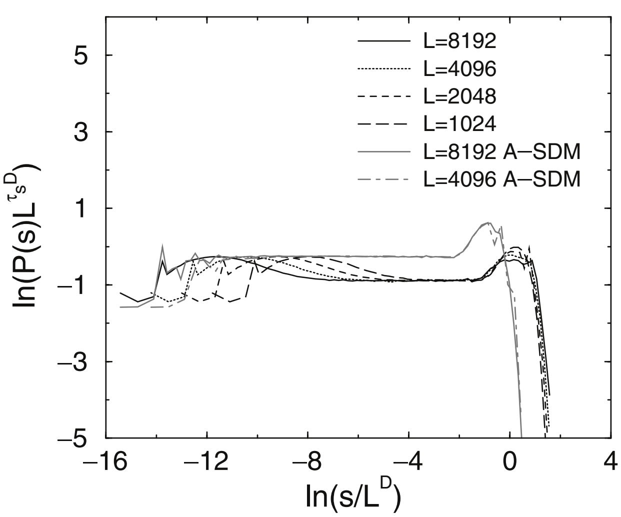

FIG. 3. Finite size scaling analysis of A-SDM and our sandpile model with $d_\perp = 1$. The values of the exponents used for the collapse are $\tau_s = 1.43$, $D = 1.75$.

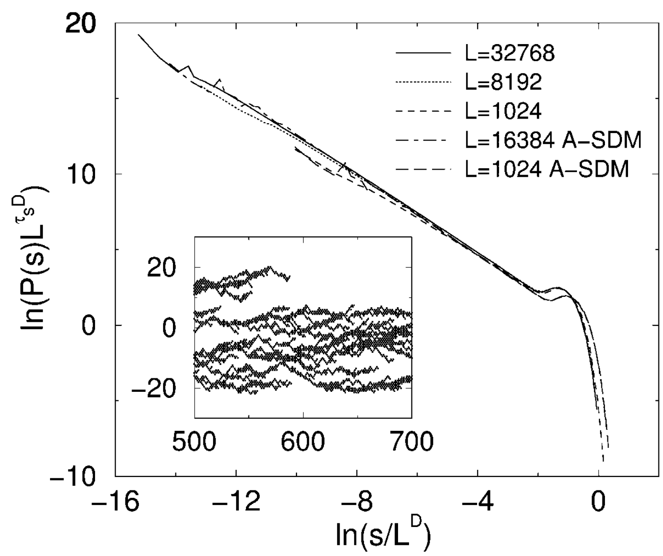

FIG. 4. Finite size scaling of the distribution of avalanche sizes for the A-SDM and our model with parallel update in $d_\perp = 0$, using $\tau_s = 1.33$ and $D = 1.5$. The inset shows the position of active sites away from the average at each time step.

The critical exponents for the single site update rule differs from those of the parallel update model. A finite size scaling analysis of the avalanche size distribution over the same range of system sizes does not collapse with the same exponents as the $d_\perp = 0$ A-SDM, but with $D = 1.1 \pm 0.02$ and $\tau_s = 1.23 \pm 0.02$. The density of occupied sites decays going into the system from the top where the grains are added. It behaves as $\rho(x_\|) \sim x_\|^{-\alpha}$, with $\alpha \approx 1$.

Some hints to understanding all of this may be found in previous analytical works. It is known that avalanches in SOC, generally, correspond to the spread of activity at an absorbing state phase transition [11–13]. They are mathematically described in the A-SDM by a variant of the Edwards-Wilkinson [14] stochastic interface equation for a dynamical height variable $h(x,t)$, where the noise amplitude is a threshold function of the local interface height [5]. In the mapping, h is the number of topplings in a given avalanche at a site, x is $x_\perp$, and t is $x_\|$. Using the analytically determined values for critical exponents from this continuum equation, good data collapse is also obtained for our non-Abelian model, where no analytic solution exists at present, in both $d_\perp = 1$ and $d_\perp = 2$. Thus, within our numerical accuracy, the critical exponents for the avalanches are the same in the Abelian and non-Abelian cases, for $d_\perp > 0$. The case $d_\perp = 0$ is special as the interface description is no longer meaningful. It appears as though the non-Abelian sandpile has organized large scale structures in such a way as to maintain the universality class, governed by a stochastic continuum equation [5], for the avalanches. Certainly, the existence of a continuum limit makes plausible a robust universality class including Abelian and non-Abelian systems, although it does not prove that the Abelian property is irrelevant to the critical behavior of the avalanches.

We emphasize that the two systems appear to be in the same universality class for the avalanches but are definitely in different universality classes for the fractal spatial structure within the metastable states. Thus, measuring properties of avalanches, alone, is not sufficient to completely determine the universality class, which must specify all the independent critical exponents, including those describing the long-range spatial and/or temporal correlations in self-organized critical systems. Returning to the concrete example of earthquakes, for instance, a model of SOC in the earthquake universality class would not only accurately describe the Gutenberg-Richter law for the distribution of earthquake sizes, but also the complex spatial and temporal scaling of the earthquake intervals and fault structure as it slowly evolves [15].

This work was supported by the EPSRC. We thank P. Bak and S. Lise for comments on the manuscript.

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