Conservation Laws, Uncertainty Relations, and Quantum Limits of Measurements

Masanao Ozawa

Center for Photonic Communication and Computing, Department of Electrical and Computer Engineering, Northwestern University, 2145 Sheridan Road, Evanston, Illinois 60208-3118 and CREST, Japan Science and Technology, Graduate School of Information Sciences, Tohoku University, Aoba-ku, Sendai, Miyagi 980-8579, Japan

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The uncertainty relation between the noise operator and the conserved quantity leads to a bound on the accuracy of general measurements. The bound extends the assertion by Wigner, Araki, and Yanase that conservation laws limit the accuracy of “repeatable,” or “nondisturbing,” measurements to general measurements, and improves the one previously obtained by Yanase for spin measurements. The bound represents an obstacle to making a small quantum computer.

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In 1952, Wigner [1] found that conservation laws put a limit on measurements of quantum mechanical observables. In 1960, Araki and Yanase [2] proved the following assertion known as the Wigner-Araki-Yanase (WAY) theorem: *Observables which do not commute with bounded additive conserved quantities have no “exact” measurements* [3]. Subsequently, Yanase [4] found a bound for the accuracy of measurements of the x component of spin in terms of the “size” of the apparatus, where the size is characterized by the mean square of the z component of the angular momentum [5]. Yanase [4] and Wigner [6] concluded from this result that in order to increase the accuracy of spin measurement one has to use a very large measuring apparatus.

In the WAY theorem, for a measurement to be “exact” the following two conditions are required to be satisfied: (i) the Born statistical formula (BSF) and (ii) the repeatability hypothesis (RH), asserting that, if an observable is measured twice in succession in a system, then we obtain the same value each time. Yanase’s bound does not assume the RH. Instead, a condition, to be referred to as Yanase’s condition, is assumed that the probe observable, the observable in the apparatus to be measured after the measuring interaction, commutes with the conserved quantity, to ensure the measurability of the probe observable [4]. Elaborating the suggestions given by Stein and Shimony [3], Ohira and Pearle [7] constructed a simple measuring interaction that satisfies the conservation law and the BSF, assuming the precise probe measurement, but does not satisfy the RH. Based on their model, Ohira and Pearle claimed that it is possible to have an accurate measurement of the spin component regardless of the size of the apparatus, if the RH is abandoned. However, their model does not satisfy Yanase’s condition, so that the problem remains as to the measurability of the probe observable.

Yanase’s argument, however, assumes a large (but of variable size) measuring apparatus having the continuous angular momentum from the beginning for technical reasons and concludes that accurate measurement requires a very large apparatus. To avoid a circular argument, a rigorous derivation without such an assumption is still demanded. Moreover, Wigner [6] pointed out the necessity for generalizing the bound to general quantum systems other than spin $1/2$ systems, as well as including all additive conservation laws.

In order to accomplish the suggested generalization, a new approach to the problem is proposed in this Letter based on the uncertainty relation between the conserved quantity and the noise operator, defined as the difference between the post-measurement probe and the measured quantity. We obtain a bound for the mean-square error of general measuring interactions imposed by any additive conservation laws without assuming the RH. This bound also clarifies the trade-off between the size and the commutativity of the noise operator with the conserved quantity, unifying the suggestion by WAY and others and the one suggested by Ohira and Pearle. For spin measurements, this bound with Yanase’s condition leads to a tight bound for the error probability of spin measurement, which improves Yanase’s bound.

Let $\mathbf{A}(\mathbf{x})$ be a measuring apparatus with macroscopic output variable $\mathbf{x}$ to measure, possibly with some error, an observable A of the *object* $\mathbf{S}$, a quantum system represented by a Hilbert space $\mathcal{H}$. The measuring interaction turns on at time t, the *time of measurement,* and turns off at time $t + \Delta t$ between the object $\mathbf{S}$ and the *probe* $\mathbf{P}$, a part of the apparatus that interacts with the object, represented by a Hilbert space $\mathcal{K}$. Denote by U the unitary operator on $\mathcal{H} \otimes \mathcal{K}$ representing the time evolution of $\mathbf{S} + \mathbf{P}$ in the time interval $(t, t + \Delta t)$.

At the time of measurement, the object is supposed to be in an unknown (vector) state ψ and the probe is supposed to be prepared in a known (vector) state ξ; all state vectors are assumed to be normalized unless stated otherwise. Thus, the composite system $\mathbf{S} + \mathbf{P}$ is in the state $\psi \otimes \xi$ at time t. Just after the measuring interaction, the probe is subjected to a local interaction with the subsequent stages of the apparatus. The last process is assumed to measure an observable M, called the *probe observable,* of the probe with arbitrary precision, and the outcome is recorded as the value of the macroscopic outcome variable $\mathbf{x}$.

In the Heisenberg picture with the original state $\psi \otimes \xi$ at time t, we shall write $A(t) = A \otimes I$, $M(t) = I \otimes M$, $A(t + \Delta t) = U^{\dagger}(A \otimes I)U$, and $M(t + \Delta t) = U^{\dagger}(I \otimes M)U$. We shall denote by "$\mathbf{x}(t) \in \Delta$" the probabilistic event that the outcome of the measurement using apparatus $\mathbf{A}(\mathbf{x})$ at time t is in an interval Δ. Since the outcome of this measurement is obtained by the measurement of the probe observable M at time $t + \Delta t$, the probability distribution of the output variable $\mathbf{x}$ is given by

$$\Pr\{\mathbf{x}(t) \in \Delta\} = \|E^{M(t+\Delta t)}(\Delta)(\psi \otimes \xi)\|^2, \tag{1}$$

where $E^{M(t+\Delta t)}(\Delta)$ stands for the spectral projection of the operator $M(t + \Delta t)$ corresponding to the interval Δ. We call the above description of the measuring process the *indirect measurement model* determined by $(\mathcal{K}, \xi, U, M)$ [8].

We say that apparatus $\mathbf{A}(\mathbf{x})$ *measures* observable A *precisely,* if $\mathbf{A}(\mathbf{x})$ satisfies the BSF for observable A,

$$\Pr\{\mathbf{x}(t) \in \Delta\} = \|E^{A}(\Delta)\psi\|^2, \tag{2}$$

on every input state ψ. Otherwise, we consider apparatus $\mathbf{A}(\mathbf{x})$ to measure observable A with some noise.

The *noise operator* N of apparatus $\mathbf{A}(\mathbf{x})$ for measuring A is defined by

$$N = M(t + \Delta t) - A(t). \tag{3}$$

The *noise* $\epsilon(\psi)$ of apparatus $\mathbf{A}(\mathbf{x})$ for measuring A on input state ψ is, then, defined by

$$\epsilon(\psi)^2 = \langle N^2 \rangle, \tag{4}$$

where $\langle \cdots \rangle$ stands for $\langle \psi \otimes \xi | \cdots | \psi \otimes \xi \rangle$. The noise $\epsilon(\psi)$ represents the root-mean-square error in the outcome of the measurement. By Eq. (4), we have

$$\epsilon(\psi)^2 \geq (\Delta N)^2, \tag{5}$$

where ΔX stands for the standard deviation of an observable X in $\psi \otimes \xi$, i.e., $(\Delta X)^2 = \langle X^2 \rangle - \langle X \rangle^2$.

We define the *noise* ϵ of apparatus $\mathbf{A}(\mathbf{x})$ to be the least upper bound of $\epsilon(\psi)$ over all possible input states ψ. One of the fundamental properties of the noise is that precise apparatuses and noiseless apparatuses are equivalent notions, as ensured by the following theorem [9]: *Apparatus* $\mathbf{A}(\mathbf{x})$ *measures observable* A *precisely if and only if* $\epsilon = 0$.

Consider the additive conservation law (ACL) for observables L_1 of the object $\mathbf{S}$ and L_2 of the probe $\mathbf{P}$, i.e.,

$$[U, L_1 \otimes I + I \otimes L_2] = 0. \tag{6}$$

In the Heisenberg picture, we shall write $L_1(t) = L_1 \otimes I$, $L_2(t) = I \otimes L_2$, $L_1(t + \Delta t) = U^{\dagger}(L_1 \otimes I)U$, and $L_2(t + \Delta t) = U^{\dagger}(I \otimes L_2)U$. The ACL, (6), can be restated as the invariance principle

$$L_1(t) + L_2(t) = L_1(t + \Delta t) + L_2(t + \Delta t). \tag{7}$$

The WAY theorem [1,2] states that, if L_1 is bounded, there is no apparatus precisely measuring A that satisfies the RH and the ACL, unless A commutes with the conserved quantity L_1. In the following argument, we shall require the ACL but abandon the RH.

Why does the conservation law limit the accuracy of measurement in general? A simple observation on the noise operator will lead to a significant interplay between the conservation law and the uncertainty relation. As we have discussed above, the measurement is precise if and only if $\langle N^2 \rangle = \|N(\psi \otimes \xi)\|^2 = 0$. If this is the case, the uncertainty relation,

$$(\Delta N)^2 \{\Delta[L_1(t) + L_2(t)]\}^2 \geq \tfrac{1}{4} |\langle [N, L_1(t) + L_2(t)] \rangle|^2, \tag{8}$$

concludes that, if the conserved quantity does not commute with the noise operator in the initial state, the conserved quantity should have infinite variance, or the precise measurement is impossible for the bounded conserved quantity.

Let us study the quantitative relations shown by the uncertainty relation, (8), in detail. Since $L_1(t)$ and $L_2(t)$ are statistically independent, the variance of their sum is the sum of their variances, i.e.,

$$\{\Delta[L_1(t) + L_2(t)]\}^2 = [\Delta L_1(t)]^2 + [\Delta L_2(t)]^2. \tag{9}$$

Since A and L_1 are in the object and M and L_2 are in the probe, we have

$$[M(t + \Delta t), L_1(t + \Delta t)] = [A(t), L_2(t)] = 0.$$

By the ACL, (7), we obtain

$$[N, L_1(t) + L_2(t)] = [M(t + \Delta t), L_2(t + \Delta t)] - [A(t), L_1(t)]. \tag{10}$$

From Eqs. (5), (8), (9), and (10), we obtain the following fundamental lower bound of the noise of apparatus $\mathbf{A}(\mathbf{x})$:

$$\epsilon(\psi)^2 \geq \frac{|\langle [M(t + \Delta t), L_2(t + \Delta t)] - [A(t), L_1(t)] \rangle|^2}{4[\Delta L_1(t)]^2 + 4[\Delta L_2(t)]^2}. \tag{11}$$

From the above lower bound, in order to attain $\epsilon = 0$ it is necessary to choose ξ, U, and M satisfying

$$\langle \xi | U^{\dagger}(I \otimes [M, L_2])U | \xi \rangle = [A, L_1]. \tag{12}$$

Stein and Shimony [3] and Ohira and Pearle [7] gave examples that actually attain $\epsilon = 0$. Does this mean that, if we abandon the RH, the ACL allows us to have a noiseless measuring apparatus regardless of the size of the apparatus? Recall that the noise ϵ is defined as the one from the measuring interaction, which quantum mechanics can analyze in detail. Thus, if we do not assume that the probe measurement is carried out precisely, the noise ϵ depends on the boundary between the probe and the rest of the apparatus. Since this boundary is rather arbitrary, it can be seen that the measuring apparatus carries out the precise measurement if and only if the noise ϵ vanishes for any boundaries. Thus, in order to show that the ACL limits the accuracy of the measuring apparatus, it suffices to show that a particular boundary leads to an inevitable noise. For this purpose, we shall consider the maximal boundary in a given apparatus. In this case, the notion of the probe is identical with a quantum mechanical description of a measuring apparatus, so that we can assume (i) the probe includes all the external sources of interactions, and (ii) the probe observable plays a role of the record. Assumption (i) is justified, since the measuring apparatus operates covariantly so that it can be used in any laboratory and at any time. Assumption (ii) is justified, since the measuring apparatus includes a record which the observer can access repeatedly. From assumption (i) we can assume that the measuring interaction satisfies the ACL. From assumption (ii) we can assume that the probe observable can be measured by another external measuring apparatus satisfying the RH. Then, the WAY theorem requires that *the probe observable should commute with the additive conserved quantities;* we call this condition Yanase's condition. Therefore, the above argument supports our hypothesis below that *in any measuring apparatus there is a boundary between the probe and the rest of the apparatus which the ACL and Yanase's condition hold.*

Now, we assume Yanase's condition,

$$[M, L_2] = 0. \tag{13}$$

In this case, the fundamental noise bound, (11), turns out to be the following form:

$$\epsilon(\psi)^2 \geq \frac{|\langle [A(t), L_1(t)] \rangle|^2}{4[\Delta L_1(t)]^2 + 4[\Delta L_2(t)]^2}. \tag{14}$$

Since the input state is unknown but the probe is prepared in a known state, the bound is optimized when the input-independent quantity $\Delta L_2(t)$ is maximized. Thus, we can conclude that *in order to decrease the noise of the apparatus, one has to increase the variance of the conserved quantity in the probe.*

Consider the case where the object **S** is a particle of spin 1/2. Let $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$ be the spin observables of **S** in the x, y, and z directions, respectively; we shall write $\alpha_i = |\hat{S}_i = \hbar/2\rangle$ and $\beta_i = |\hat{S}_i = -\hbar/2\rangle$ for $i = x, y, z$. In what follows, we shall optimize the noise ϵ of apparatus $\mathbf{A}(\mathbf{x})$ for measuring the x component of the spin of particle **S**, under the following constraints: (i) The measuring interaction preserves the z component of the total angular momentum, i.e.,

$$[U, \hat{S}_z + \hat{L}_z] = 0, \tag{15}$$

where $\hat{L}_z$ is the z component of the angular momentum of probe **P**, and (ii) the probe observable M commutes with the conserved quantity, i.e.,

$$[M, \hat{L}_z] = 0. \tag{16}$$

By the optimization it is meant, here, to obtain the saddle point in which the bound is maximized by the object state and minimized by the probe state. From the above constraints, Eq. (14) holds for $A = \hat{S}_x$, $L_1 = \hat{S}_z$, and $L_2 = \hat{L}_z$. By the relation $[A, L_1] = [\hat{S}_x, \hat{S}_z] = -i\hbar\hat{S}_y$, we obtain the following bound for the noise:

$$\epsilon(\psi)^2 \geq \frac{\hbar^2 \langle \hat{S}_y(t) \rangle^2}{4[\Delta \hat{S}_z(t)]^2 + 4[\Delta \hat{L}_z(t)]^2}. \tag{17}$$

For apparatuses with large $[\Delta \hat{L}_z(t)]^2$, the optimal bound achieves when the numerator of the right-hand side of Eq. (17) is maximized. This is achieved by $\psi = \alpha_y$, for instance, in which we have $\langle \hat{S}_y(t) \rangle = \Delta \hat{S}_z(t) = \hbar/2$. In this case, we have the optimal bound as follows:

$$\epsilon^2 \geq \epsilon(\alpha_y)^2 \geq \frac{\hbar^2}{4 + 16[\Delta \hat{m}_z]^2}, \tag{18}$$

where $\hat{m}_z$ is the initial angular momentum normalized by $\hbar$, i.e., $\hat{m}_z = \hat{L}(t)_z/\hbar$. If $\Delta \hat{m}_z$ is not large enough, the right-hand side of Eq. (18) may not be optimal; however, Eq. (18) still gives a correct lower bound, since our derivation uses no approximation.

For spin 1/2 measurements, the mean-square error is considered to be the $\hbar^2$ times the error probability, and, hence, we should define the error probability $P_e(\psi)$ by

$$P_e(\psi) = \frac{\epsilon(\psi)^2}{\hbar^2}. \tag{19}$$

Then, the maximum error probability P_e is bounded by

$$P_e \geq P_e(\alpha_y) \geq \frac{1}{4 + 16[\Delta \hat{m}_z]^2}. \tag{20}$$

For the engineering of microscopic information processors such as quantum logic gates [10], this bound is considered to be a serious obstacle to realizing small and accurate quantum devices.

In addition to the formulation discussed above, Yanase [4] and Wigner [6] considered the measuring interaction with the following form:

$$U(\alpha_x \otimes \xi) = \alpha_x \otimes \xi^+ + \beta_x \otimes \eta^+, \tag{21a}$$

$$U(\beta_x \otimes \xi) = \beta_x \otimes \xi^- + \alpha_x \otimes \eta^-. \tag{21b}$$

The states $\xi^{\pm}$ and $\eta^{\pm}$ are not normalized. The states $\xi^{\pm}$ are assumed to be eigenstates of the observable M satisfying

$$M\xi^{\pm} = \pm\frac{\hbar}{2}\,\xi^{\pm}. \tag{22}$$

The problem is to find a lower bound of the sum of the two "unsuccessful probabilities" $\|\eta^+\|^2$ and $\|\eta^-\|^2$,

$$\epsilon_Y^2 = \|\eta^+\|^2 + \|\eta^-\|^2, \tag{23}$$

to show a trade-off with the size of the apparatus characterized by the mean square, $\langle \hat{m}_z^2 \rangle$, of the normalized angular momentum.

Under these, and the additional technical assumptions that ϵ_Y is very small and that $\langle \hat{m}_z^2 \rangle$ is so large that the eigenvalues of $\hat{m}_z$ can be treated as a continuous parameter, Yanase [4] obtained the following lower bound:

$$\epsilon_Y^2 > \frac{1}{8\langle \hat{m}_z^2 \rangle}. \tag{24}$$

Later, Ghirardi *et al.* [5] derived the same bound for rotationally invariant interactions without continuous parameter approximation.

In what follows, we shall obtain a tighter bound for ϵ_Y^2 without any approximation. For this purpose, we shall show the relation

$$\epsilon_Y^2 \geq \frac{2}{\hbar^2}\,\epsilon(\alpha_y)^2 = 2P_e(\alpha_y). \tag{25}$$

The proof runs as follows. Easy computations show

$$UN(\alpha_x \otimes \xi) = \beta_x \otimes \left(M - \frac{\hbar}{2} I\right)\eta^+, \tag{26a}$$

$$UN(\beta_x \otimes \xi) = \alpha_x \otimes \left(M + \frac{\hbar}{2} I\right)\eta^-. \tag{26b}$$

By the relation $2\alpha_y = (1 + i)\alpha_x + (1 - i)\beta_x$, we have

$$\begin{aligned}\epsilon(\alpha_y)^2 = \ &\|UN(\alpha_y \otimes \xi)\|^2 = \frac{1}{2}\left\| \left(M - \frac{\hbar}{2} I\right)\eta^+ \right\|^2 \\ &+ \frac{1}{2}\left\| \left(M + \frac{\hbar}{2} I\right)\eta^- \right\|^2 \\ &\leq \frac{\hbar^2}{2}\|\eta^+\|^2 + \frac{\hbar^2}{2}\|\eta^-\|^2 = \frac{\hbar^2}{2}\,\epsilon_Y^2 .\end{aligned}$$

Thus, we obtain Eq. (25). By combining relations (20) and (25), we conclude

$$\epsilon_Y^2 \geq \frac{1}{2 + 8(\Delta m_z)^2}\,. \tag{27}$$

Under the conditions (i) $1 \ll (\Delta m_x)^2$ and (ii) $\langle \hat{m}_z \rangle \approx 0$, Yanase's bound, (24), turns out to be a good approximation for the rigorous bound, (27), and otherwise the new bound is tighter.

In order to show that the bound (20) typically vanishes for macroscopic apparatuses, we assume that the probe is a three-dimensional isotropic harmonic oscillator in a coherent state. Let $|\alpha\rangle$ and $|\beta\rangle$ be the coherent states quantized along the x and y axes, respectively. Then from Ref. [11] we have

$$(\Delta \hat{m}_z)^2 = |\alpha|^2 + |\beta|^2, \tag{28}$$

and, hence, the optimal bound turns to

$$P_e \geq \frac{1}{4 + 16|\alpha|^2 + 16|\beta|^2}\,. \tag{29}$$

If the probe is a macroscopic harmonic oscillator, we have $|\alpha|^2, |\beta|^2 \gg 1$, and, hence, the error probability P_e can be arbitrarily small.

We have obtained a bound for the accuracy of general measurements imposed from conservation laws and uncertainty relations. This bound shows that, in order to make a precise measurement, the probe is required to have very large variance of the conserved quantity, as long as the probe can be observed repeatedly. If the apparatus is macroscopic, this bound poses no serious limit. However, for quantum information processing, measuring interactions occur between qubits, which can have only a small amount of conserved quantities. The relevance of this bound with the fundamental limit of quantum information processing will be worth further investigations.

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