Nonequilibrium Phase Transitions in Directed Small-World Networks

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Many social, biological, and economic systems can be approached by complex networks of interacting units. In many of these systems relations are *directed* in the sense that links act only in one direction (outwards or inwards). We investigate the effect of directed links on the behavior of a simple spin-like model evolving on a small-world network. This model may describe for instance the dynamics of public opinion in social influence networks. We show that directed networks may lead to a highly nontrivial phase diagram including first- and second-order phase transitions out of equilibrium.

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Complex networks have recently attracted an increasing interest among physicists, the main reason being that they seem to be exceedingly simple model systems of complex behavior in real world networks [1,2], including chemical reaction networks [3], food webs [4–6], the Internet [7,8] and the World Wide Web [9], metabolic [10] and protein networks [11], scientific collaboration networks [12], etc. The hope is that the ideas and techniques, developed in the past fifty years in the field of statistical physics to deal with cooperative phenomena in many body systems, may be useful to understand emergent complex behavior in systems outside the traditional realm of physics. In particular, small-world (SW) networks, recently introduced by Watts and Strogatz [13], have been very much studied because they constitute an interesting attempt to translate the complex topology of social, economic, and physical networks into a simple model. SW networks result from randomly replacing a fraction p of links of a d-dimensional regular lattice with new random links. As a result of this random rewiring, SW networks interpolate between the two limiting cases of a regular lattice ($p = 0$) and completely random graphs ($p = 1$). Studies of real network data have shown that SW-like topologies are found in situations as diverse as the network of movie actors' collaboration, the electric power grid of Southern California, the network of world airports, the acquaintance network of Mormons, etc. [1,2].

Many topological properties of the SW model have recently been investigated, as for instance, the shortest-path distance and clustering coefficient [13,14], the crossover from regular to SW behavior occurring at $p = 0$ [15], a mean-field solution [16], and percolation on SW networks [17], among others. Specifically, SW models are expected to play an important role in understanding the interplay between the underlying disordered network and the dynamics of many social or economic processes, such as distribution of wealth, disease spreading, transmission of cultural traits, and formation of public opinion [1].

In the language of social network analysis [18], sites are referred to as *actors*. Actors may represent individuals, companies, airports, countries, etc., depending on the social or economic process we are interested in. Actors are linked to one another by a relational, social, or physical *tie* as, for instance, friendship, business transactions, flight connections, kinship, or scientific collaboration, among many others. Some of those relational links are *symmetric*, in the sense that if Alice is tied to Bob, then Bob must also be tied to Alice, as occurs, for instance, in the authorship of scientific papers. However, many other networks are *directed* and exhibit links that are definitely *asymmetric*, as for instance, in the case of networks of the import and the export of goods, World Wide Web page links, lending transactions, food webs, cultural influences, etc. In directed networks then, when Alice is tied to Bob, Bob may not be linked to Alice but to someone else instead. Asymmetric synaptic strengths have already been shown to be very important in trying to describe the process of learning in realistic neural network model approaches to brain function [19,20].

Several models have recently been studied in order to understand the effect of SW topology on classical systems such as the Ising model [14] or the spread of infections and epidemics [21,22]. Such simple models are expected to capture the essential features of the more complicated processes taking place on real networks. However, as mentioned earlier, many social, commercial, or biological relations are asymmetric and the following question naturally arises: What is the effect of directed links on a simple model that evolves on the network?

In this Letter, we investigate the effect of directed SW topology on the behavior of a simple model. In case only undirected links are used, our model becomes identical to the classical Ising model on a standard (undirected) SW network [14]. This allows us to study, in a systematic way, the effect of directed ties on this classical model. We find that the existence of directed links completely changes the behavior of the system from mean-field behavior (for undirected networks) to a highly nontrivial and rich phase diagram in the case of directed networks. By means of extensive numerical simulations, we find that, for rewiring probabilities in the range $0 < p < p_c$, the model exhibits a line of continuous phase transitions from an ordered to a disordered state. Those phase transitions occur at a critical value of the temperature $T_c(p)$, which depends on p.

However, for higher disorder densities $p_c < p \leq 1$, the phase transition becomes first order. Our results show that, in order to model biological, social, or economic processes on complex networks, it is crucial to take into account the character, directed or undirected, of the corresponding relational links.

The model.—We have studied directed networks in $d = 1$ and 2. For simplicity, we focus here on $d = 2$, and further results in $d = 1$ will be published elsewhere. In order to construct a directed SW network, we start from a two-dimensional square lattice consisting of sites linked to their four nearest neighbors by both outgoing and incoming links. Then, with probability p, we reconnect nearest-neighbor outgoing links to a different site chosen at random. After repeating this process for every outgoing link, we are left with a network with a density p of SW directed links, as shown in Fig. 1. Note that by this procedure every site will have exactly four outgoing links and a varying (random) number of incoming links. Generalization to higher dimensions is straightforward.

Adopting social network nomenclature, actors are then placed at the network sites. Any given actor is connected by four outgoing links to other actors, which we call *mates*. We allow every actor to be in one of two possible states, so that, at any given time, the state of an actor is described by a binary spin-like variable $s_i \in \{+1, -1\}$. Depending on the state of their mates, an actor may change its state according to a majority (ferromagnetic) rule: Actors prefer to be in the same state as their mates. In order to implement this, we introduce the payoff function:

$$G(i) = 2s_i \sum_{\text{mates of } i} s_j , \qquad (1)$$

where the sum is carried out over the four mates of actor i. Note that this payoff function is positive whenever

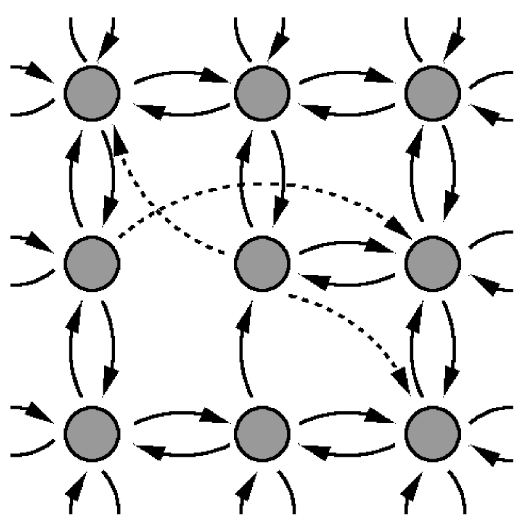

FIG. 1. Sketch of a directed small-world network constructed from a square regular lattice in $d = 2$. For the sake of clarity, only a few links have been reconnected. Arrows indicate the direction of the corresponding link. Dotted lines represent rewired links. Note that every site always has four outgoing links.

s_i points in the same direction as the majority of its four mates. External noise is included to allow some degree of randomness in the time evolution by means of a temperaturelike parameter, T, which we shall call temperature for short from now on. For a given value of the external temperature, the update of the model is then performed as follows: At each time step, an actor (network site) is randomly chosen and its corresponding payoff function $G(i)$ is computed according to Eq. (1). If $G(i) < 0$, actor i is opposing its mates' majority and the change $s_i \to -s_i$ is accepted. Unfavorable changes, i.e., when $G(i) > 0$, are accepted with probability $\exp[-G(i)/T]$, which depends on temperature in the usual fashion.

Concerning the physics of the above defined model, there are two interesting points that should be explicitly mentioned. On the one hand, the model is nonequilibrium since detailed balance is not satisfied. On the other hand, the model is not simply the asymmetric counterpart of the Ising model, since the payoff function $G(i)$ in Eq. (1) does not include the corresponding interaction terms coming from the ingoing links [needed in order to identify $G(i)$ with the energy change after a spin update in the asymmetric Ising model]. In fact, one can easily see that the payoff function $G(i)$ of our model cannot be written as a variation of any Hamiltonian. However, we would like to remark that, if only symmetric links are allowed, our model becomes exactly equal to the (equilibrium) Ising model in an undirected SW network that was studied in Ref. [14]. This can be seen by a simple comparison of the payoff function Eq. (1) with the change of energy after a spin update in the standard (symmetric) Ising ferromagnet. In this case, it is known that the system presents mean-field behavior for any value of the disorder $p > 0$ [14].

Results.—We have carried out extensive numerical simulations of the model for different values of the density of SW directed ties p and temperature. Our results are qualitatively the same for directed networks generated from regular lattices in $d = 1$ and 2. In the following, we focus on $d = 2$. We have simulated the model in directed SW networks generated from $L \times L$ square lattices for sizes ranging from $L = 8$ to 100 and different rewiring probabilities $p \in [0, 1]$. The system is left to evolve until, after some transient, a stationary nonequilibrium state is reached. The stationary state can be described by the appropriate order parameter, which can be defined in a natural way by means of the "magnetization" per site:

$$m = \frac{1}{L^2} \sum_{i=1}^{L^2} s_i . \qquad (2)$$

We find that the system becomes ordered, i.e., $\langle |m| \rangle \neq 0$, below a critical temperature $T_c(p)$, so that most actors are, in average, in the same state. In Fig. 2 the average absolute value of the order parameter is plotted vs temperature for two different values of the disorder $p = 0.1$ and 0.9, calculated in systems of different sizes. For every system size L^2, results were averaged over both, ten runs of the dynamics for each network realization and

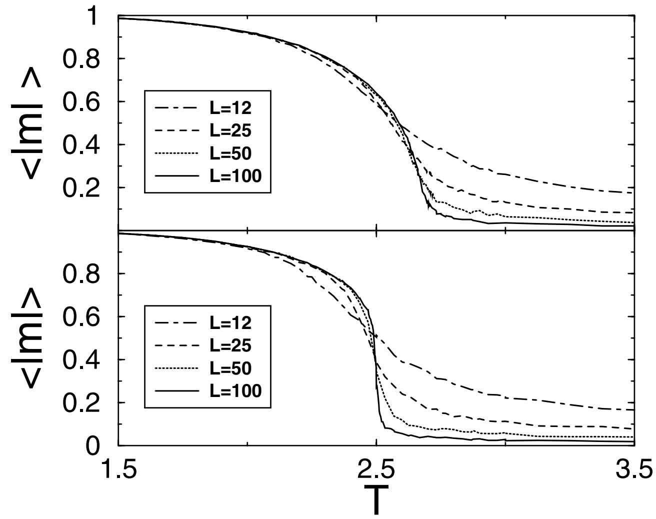

FIG. 2. Order parameter vs T for different system sizes. For $p = 0.1$ (top panel), the transition is continuous. For a higher disorder density, $p = 0.9$, the transition becomes first order (bottom panel).

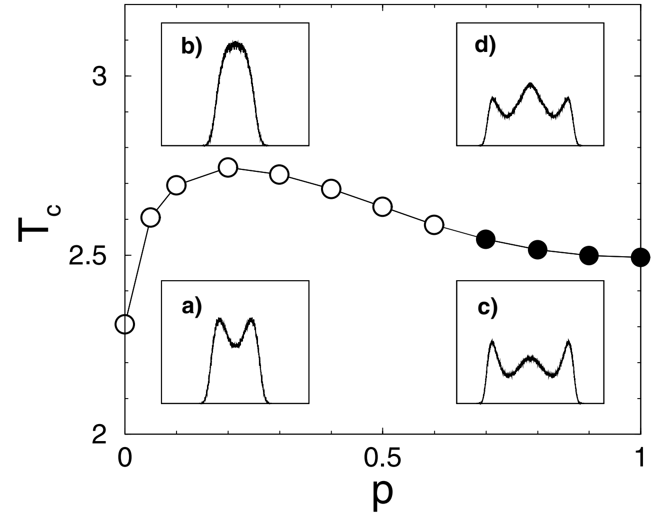

FIG. 3. Phase diagram of the model. The system is in the ordered state below the line. Points are numerical determinations of the critical temperatures $T_c(p)$ for different degrees of disorder. The transition is continuous for small values of p (circles), while it becomes discontinuous for p larger than $p_c \approx 0.65$ (filled circles). The insets show the PDFs of m for a) $p = 0.1$, $T = 2.68$; b) $p = 0.1$, $T = 2.70$; c) $p = 0.9$, $T = 2.498$; d) $p = 0.9$, $T = 2.500$. Simulations were performed in a 100×100 sites network.

n different realizations of the network, in such a way that $n \times L^2 \approx 1.5 \times 10^5$. Figure 2 shows that the order-disorder transition is continuous (top panel) for a low disorder density, while it becomes discontinuous for a higher concentration of directed SW links (bottom panel). A more systematic study of the phase diagram, as shown in Fig. 3, reveals that there is a line of continuous phase transitions for disorder densities below some critical value p_c. Very interestingly, the transition becomes first order above p_c, indicating that there exists a *nonequilibrium tricritical point* at p_c. We estimate p_c to be roughly at $p_c = 0.65(5)$. The character, continuous or discontinuous, of the phase transition is better realized when looking at the probability density function (PDF) of the order parameter. For the sake of illustration, the insets of Fig. 3 show typical PDFs at points of the phase diagram (p, T), all near the critical line. From those PDFs, one can see that the phase transition is second order from Figs. 3a to 3b, in the region $p < p_c$. In contrast, for $p > p_c$ the transition is discontinuous, from Fig. 3c to 3d. The most probable values of m, which correspond to the two equally highest symmetric peaks in Fig. 3c, become unstable in favor of $m = 0$ as the transition line is crossed towards Fig. 3d. The transition occurs in such a way that the order parameter exhibits a finite jump at the critical line.

The critical behavior of the model in the region $p < p_c$, where transitions are continuous, can be studied in detail. We find that the order parameter exhibits finite-size scaling with exponents that depend on the disorder density p. Close to the critical point, $|t| \to 0$, we have $\langle |m| \rangle \sim |t|^\beta$, where $t = (T - T_c)/T_c$ is the reduced temperature. At the critical point, $t = 0$, the order parameter scales with system size as $\langle |m| \rangle \sim L^{-\beta/\nu}$, where ν is the correlation length exponent. Figure 4 displays the behavior of $\langle |m| \rangle$ vs L for two disorder densities $p = 0.1$ and 0.5 below $p_c \approx 0.65$. Only for $T = T_c(p)$, a power law is obtained and the slope of the straight line in a log-log plot gives an estimation of the ratio β/ν between critical exponents. From Fig. 4 we obtain that, for $p = 0.1$ and $p = 0.5$, $\beta/\nu = 0.53(2)$ and 0.40(3), respectively. Moreover, from data collapse analysis (not shown) at the corresponding $T_c(p)$, we have $\beta = 0.50(3)$, $\nu = 0.94(6)$ and $\beta = 0.30(3)$, $\nu = 0.80(3)$ for $p = 0.1$ and $p = 0.5$, respectively. These critical exponents are different from both mean-field ($\beta = \nu = 1/2$) and exact values ($\beta = 1/8$ and $\nu = 1$) for the Ising model in $d = 2$.

Conclusions.—Many social, economic, and biological networks in the real world exhibit directed ties or relations. This may be modeled by including directed links in the corresponding complex network model. In addition, spin-like models, borrowed from statistical physics, have recently been proposed as toy models to understand some social processes, as for instance, conflict vs cooperation among coalitions [23,24] or formation of cultural domains [25]. We claim that the directedness of the network may strongly affect the behavior of simple processes evolving on complex networks. We studied the effect of a directed small-world topology on a very simple spin model. This model may be thought of as a simple description of the dynamics of public opinion in real social networks, where individuals are represented by actors, social influence by directed links, and the spin variables mimic individuals' state of opinion. Our model becomes equal to the Ising model when all links are undirected. From numerical simulations we showed that, when directed links exist, the phase diagram of the model is nontrivial. We found that

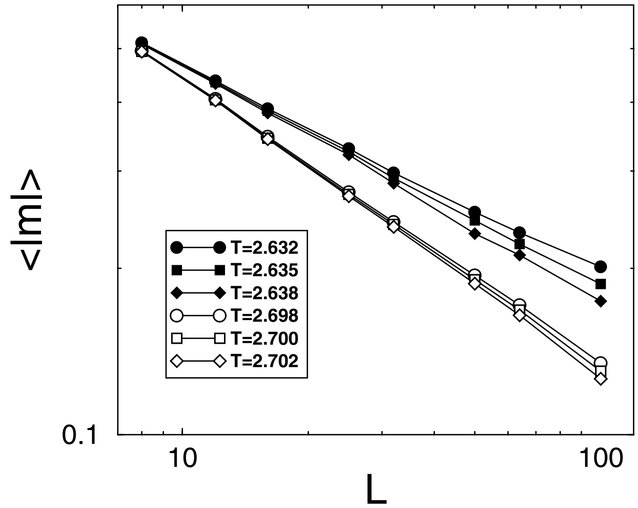

FIG. 4. Finite-size scaling of the order parameter for $p = 0.1$ (hollow symbols) and $p = 0.5$ (filled symbols), both below p_c. Power-law behavior is obtained for $T = 2.700(2)$ and $T = 2.635(3)$ for $p = 0.1$ (squares) and $p = 0.5$ (filled squares), respectively.

the system exhibits continuous phase transitions for disorder densities below a critical threshold $p_c \approx 0.65$. For stronger disorder, the transition is first order. At this stage we can only speculate that the competition among weekly coupled clusters may be related to the existence of first-order transitions.

We believe that the effect of directed links may be relevant in other types of disordered networks, such as free-scale networks, and different dynamical models. In trying to model real systems, directed links may play an important and unforeseen role.

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