# Antiferromagnetism from Phase Disordering of a $d$-Wave Superconductor

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The unbinding of vortex defects in the superconducting condensate with $d$-wave symmetry at $T = 0$ is shown to lead to the insulator with incommensurate spin-density-wave order. The transition is similar to the spontaneous generation of the chiral mass in the three-dimensional quantum electrodynamics. A possible relation to recent experiments on underdoped cuprates is discussed.

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A common feature of all high-temperature superconductors is that undoped they are Mott insulators with antiferromagnetic order [1]. The central theme of the theories of cuprate superconductivity has therefore been to establish the connection between the insulating and the superconducting phases. Most of the work followed the usual route that suggests starting from the nonsuperconducting, in this case, Mott insulating phase, and trying to understand how it becomes superconducting. This approach was spectacularly successful for the conventional (low-$T_c$) superconductors, in part because the nonsuperconducting phase was a well understood metallic Fermi liquid. In cuprates, however, one does not enjoy this luxury, and the Mott insulator is strongly interacting and notoriously resistant to simple theoretical understanding. This suggests one should look for alternative points of view that may be better adapted to the problem at hand. Since experimentally the superconducting phase seems to be a rather standard BCS-like $d$-wave state, one strategy would be to take this as a vantage point for further exploration of the cuprates phase diagram [2,3]. Particularly interesting is the underdoped region, where experiments show a large pseudogap for spin excitations, and the superconductor-insulator transition at low temperatures.

In this Letter I subscribe to the dual approach advocated above and show that the $d$-wave superconducting state (dSC) at $T = 0$ has an instability towards the insulator with the incommensurate spin density wave (SDW) order. Using the Franz-Tešanović transformation I derive the low-energy theory for the coupled system of $d$-wave quasiparticles and fluctuating vortices. Upon integration over vortices the theory takes the form of the (anisotropic) $(2 + 1)$-dimensional quantum electrodynamics (QED3) for two Dirac four-component spinors, which are related to the nodal quasiparticles by a singular gauge transformation, and are minimally coupled to the transverse gauge field [3]. First, I show that the role of the coupling constant (or the "charge") in this gauge theory is at $T \neq 0$ played by the thermodynamic fugacity of the vortex system. In the superconducting phase the charge is therefore zero, vortices are bound into pairs, and the gauge field is decoupled from the fermions. In the nonsuperconducting phase, on the other hand, the fugacity is finite, and the gauge field now mediates a long-range interaction between the Dirac fermions. The main result is that, at $T = 0$ where the role of fugacity is played by the condensate of vortex loops, this interaction leads to an instability towards the incommensurate SDW order, through a condensed-matter equivalent of the chiral symmetry breaking phenomenon [4]. The $T = 0$ transition from the dSC into the SDW may be understood therefore as an instability of the gapless nodal fermionic excitation in the presence of free topological defects towards the formation of bound states. Possible connections between recent neutron scattering, angle-resolved photoemission spectroscopy (ARPES), and scanning tunneling microscopy (STM) experiments are discussed in light of this result.

What follows rests on two postulates: (1) that there is a $d$-wave superconducting state in the phase diagram with sharp gapless quasiparticle excitations, and (2) that the amplitude of the superconducting order parameter may be assumed finite and inert much below the high pseudogap temperature $T^*$, so that the only other relevant excitations are the topological defects in its phase (vortices and antivortices at $T \neq 0$, or vortex loops at $T = 0$). The first postulate is supported by the microwave [5] and the ARPES experiments [6], and the second by the measurements of the frequency dependent conductivity [7] and the Nernst effect in the pseudogap regime [8]. I begin by constructing the continuum, low-energy theory for the nodal quasiparticles in the $d$-wave state, using a different representation than in [2,3]. The quasiparticle Hamiltonian is

$$H_{qp} = T \sum_{\vec{k},\sigma,\omega_n} [(i\omega_n - \xi_{\vec{k}})c_\sigma^\dagger(\vec{k},\omega_n)c_\sigma(\vec{k},\omega_n) - \Delta(\vec{k})c_\sigma^\dagger(\vec{k},\omega_n)c_{-\sigma}^\dagger(-\vec{k},-\omega_n) + \text{c.c.}], \tag{1}$$

where $\Delta(\vec{k})$ has the usual $d$-wave symmetry, and two spatial dimensions (2D) are assumed. $c$ and $c^\dagger$ are the electron operators, $\sigma = \pm$ labels spin, and $\omega_n$ are the fermionic Matsubara frequencies. In my units $h = c = e = 1$. Next, introduce *two four-component* Dirac spinors

$$\Psi^\dagger_{1(2)}(\vec{q},\omega_n) = [c^\dagger_+(\vec{k},\omega_n), c_-(-\vec{k},-\omega_n), c^\dagger_+(\vec{k}-\vec{Q}_{1(2)},\omega_n), c_-(-\vec{k}+\vec{Q}_{1(2)},-\omega_n)], \quad (2)$$

where $\vec{Q}_{1(2)} = 2\vec{K}_{1(2)}$ is the wave vector that connects the nodes within the diagonal pair 1(2). For the spinor 1, $\vec{k} = \vec{K}_1 + \vec{q}$, with $|\vec{q}| \ll |\vec{K}_1|$ (see Fig. 1), and analogously for the second pair. One has $\xi_{\vec{k}} = \xi_{-\vec{k}}$, and *near* the nodes, $\xi_{\vec{k}} = -\xi_{\vec{k}-\vec{Q}_{1(2)}}$, and $\Delta_{\vec{k}} = -\Delta_{\vec{k}-\vec{Q}_{1(2)}}$, for $\vec{k} \approx \vec{K}_{1(2)}$. Retaining only the low-energy modes in (1), and linearizing the spectrum as $\xi_{\vec{k}} = v_f q_x$ and $\Delta_{\vec{k}} = v_\Delta q_y$, one arrives at the continuum field theory

$$S[\Psi] = \int d^2\vec{r} \int_0^\beta d\tau\, \bar{\Psi}_1[\gamma_0\partial_\tau + \gamma_1 v_f \partial_x + \gamma_2 v_\Delta \partial_y]\Psi_1 + (1 \to 2, x \to y, y \to x), \quad (3)$$

where $\bar{\Psi} = \Psi\gamma_0$, and the matrices $\gamma_0 = \sigma_1 \otimes I$, $\gamma_1 = \sigma_2 \otimes \sigma_3$, and $\gamma_2 = -\sigma_2 \otimes \sigma_1$ satisfy the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu,\nu}$. Here $\vec{\sigma}$ are the Pauli matrices, and the coordinate system has been rotated as in Fig. 1.

Next, assume that the transition out of the dSC in the underdoped regime is due to the unbinding of the topological defects. This raises a rather nontrivial question of how to properly couple the vortex degrees of freedom to quasiparticles [2]. Fortunately, this has recently been elegantly solved by Franz and Tešanović [3,9]. Their idea is to split the phase of the order parameter $\Delta(\vec{r},\tau) = |\Delta| \exp i[\phi_s(\vec{r},\tau) + \phi_r(\vec{r},\tau)]$, where $\phi_{s(r)}$ is the singular (regular) part of the phase in the presence of vortices, into two contributions, $\phi_s + \phi_r = \phi_A + \phi_B$, with

$$\nabla \times \nabla\phi_A(\vec{r},\tau) = 2\pi \sum_{iA=1}^{N_A} q_{iA}\delta[\vec{r} - \vec{r}_{iA}(\tau)], \quad (4)$$

where $q_{iA} = \pm 1$ is the unit vorticity of a $(hc/2e)$ vortex (antivortex) defect, tracked by its coordinate in the imaginary time $\vec{r}_i(\tau)$, and analogously for vortices in $B$. Here $\nabla = (\partial_\tau, \partial_x, \partial_y)$. The division of vortices into two groups $A$ and $B$ is at this point arbitrary, and $\phi_r$ is to be equally split between $\phi_A$ and $\phi_B$. By making the singular gauge transformation in (1) from electrons into electrically *neutral* fermions $c_{+(-)} \to c_{+(-)} \exp i\phi_{A(B)}$, one immediately discovers that there is a hidden gauge field in the problem, $a_\mu = \frac{1}{2}\partial_\mu(\phi_A - \phi_B)$, $\mu = 0,1,2$, that enters the theory (3) via *minimal* coupling $\partial_\mu \to \partial_\mu - ia_\mu$. The regular part of the phase $\phi_r$ cancels in $\vec{a}$, which is entirely due to vortices. $\phi_r$ is contained in the second Doppler gauge field, $v_\mu = \frac{1}{2}\partial_\mu(\phi_A + \phi_B)$, which enters the theory for the neutral fermions precisely as the true electromagnetic gauge field would. Gauge invariance protects $\vec{a}$ from becoming massive from the integration over fermions, while such a protectorate does not exist for $\vec{v}$. Power counting implies then that the coupling of fermions to $\vec{v}$ is irrelevant, and may and will therefore be dropped hereafter.

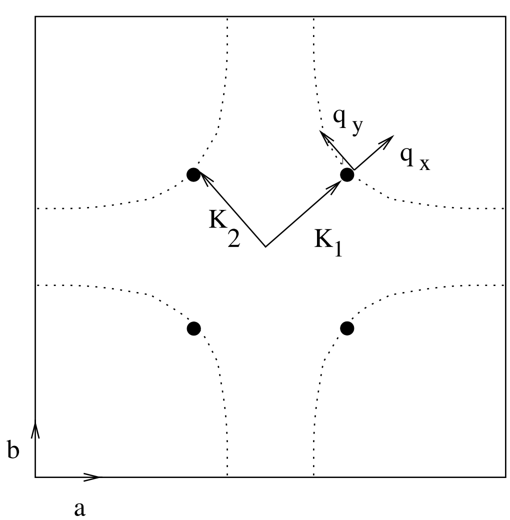


FIG. 1. The wave vectors $\vec{K}_1$, $\vec{K}_2$, and $\vec{q}$.

It was argued in [3] that although $\vec{a}$ cannot become massive from fermions, it should be massive if vortices are bound. Next I present a simple but a rigorous derivation of the dynamics of the gauge field $\vec{a}$ at $T \neq 0$, where one can get away with the neglect of the quantum fluctuations, which supports this insight. Assume a collection of $N_+(N_-)$ vortices (antivortices) at the positions $\{\vec{r}_i\}$. The energy of the (classical) vortex system is

$$H_v = \frac{1}{2}\sum_{i=1}^{N} q_i q_j v(\vec{r}_i - \vec{r}_j), \quad (5)$$

where $v(\vec{r}) \approx -\ln|\vec{r}|$, at large distances, and $N = N^+ + N^-$. The partition function of the coupled system of quasiparticles and vortices can be written as

$$Z = \int D[\Psi]\, e^{-S[\Psi,\vec{a}]} Z_v, \quad (6)$$

where $Z_v$ is the grand-canonical classical partition function of the vortex system (2D Coulomb plasma)

$$Z_v = \sum_{N_A^-, N_A^+, N_B^-, N_B^+} \frac{y^N \int \prod_{i=1}^N d\vec{r}_i\, e^{-H_v/T}}{2^N N_A^+!\, N_B^+!\, N_A^-!\, N_B^-!}, \quad (7)$$

where $N^{+(-)} = N_A^{+(-)} + N_B^{+(-)}$, and $y$ is the bare vortex fugacity. To preserve the $\sigma \to -\sigma$ symmetry in the original Hamiltonian in $Z_v$ I average over *all* possible divisions of vortices and antivortices into groups $A$ and $B$. This ensures that on average there is an equal number of vortices (and antivortices) in both groups. Next, introduce the vorticity densities in $Z_v$ by inserting the unity

$$1 = \int D[\rho_A]\, \delta\left[\rho_A(\vec{r}) - \sum_{i=1}^{N_A} q_{iA}\delta(\vec{r} - \vec{r}_{iA})\right], \quad (8)$$

and similarly for $B$. The gauge field then becomes

$$[\nabla \times \vec{a}(\vec{r})]_\tau = 2\pi[\rho_A(\vec{r}) - \rho_B(\vec{r})], \quad (9)$$

in the transverse gauge $\nabla \cdot \vec{a} = 0$. Subindex $\tau$ denotes the $\tau$ component. $\vec{v}$ is defined the same way except with the plus sign between $\rho_A$ and $\rho_B$. Performing then the Gaussian integrations over $\rho_A$, $\rho_B$, and $\vec{v}$, and the summations in Eq. (7) *exactly,* yields $Z_v = \int D[\Phi_+, \Phi_-, \vec{a}] \exp(-S_v[\Phi_+, \Phi_-, \vec{a}])$, with

$$S_v[\Phi_+, \Phi_-, \vec{a}] = \int d^2\vec{r} \left[ 2T(\nabla\Phi_+(\vec{r}))^2 + \frac{i}{2\pi} \Phi_-(\vec{r}) (\nabla \times \vec{a}(\vec{r}))_\tau - 2y \cos(\Phi_+(\vec{r})) \cos(\Phi_-(\vec{r})) \right]. \quad (10)$$

Real fields $\Phi_\pm = \Phi_A \pm \Phi_B$ are the Lagrange multipliers introduced to enforce the constraints in Eq. (8) [10].

The partition function of the coupled system of $d$-wave quasiparticles and vortices at $T \neq 0$ is therefore $Z = \int D[\Psi, \Phi_+, \Phi_-, \vec{a}] \exp(-S[\Psi, \vec{a}] - S_v[\Phi_+, \Phi_-, \vec{a}])$ with

$$S[\Psi, \vec{a}] = \sum_{i=1}^{F} \int d^2\vec{r} \int_0^\beta d\tau\, \bar{\Psi}_i \gamma_\mu (\partial_\mu - i a_\mu) \Psi_i , \quad (11)$$

with $F = 2$, and the $x \leftrightarrow y$ exchange of the coordinates for the $i = 2$ component is assumed. I have also set $v_f = v_\Delta = 1$ here for simplicity. The Dirac field $\Psi$ represents the neutral (gauge-transformed) fermions, and $S_v$ is given by Eq. (10). This is my first result. It has several remarkable features. First, if one turns off the coupling to fermions (by taking, formally, the quenched limit $F = 0$), the integration over $\vec{a}$ in (10) simply enforces $\Phi_- \equiv 0$. $S_v$ reduces then to the standard sine-Gordon theory, which is known to provide the correct description of the Kosterlitz-Thouless transition [11]. More importantly for our purposes, for $F = 0$ one also finds

$$\langle [\nabla \times \vec{a}(\vec{r})]_\tau [\nabla \times \vec{a}(\vec{r}\,')]_\tau \rangle = \langle y \rangle \delta(\vec{r} - \vec{r}\,'), \quad (12)$$

where $\langle y \rangle = y(2\pi)^2 \langle \exp(i\Phi_+) \rangle$, with the average to be taken over $S_v$ with $\Phi_- \equiv 0$. $\langle y \rangle$ may be recognized as the *thermodynamic,* or the renormalized, fugacity of the vortex system [11]. The integration over the fields $\Phi_+$ and $\Phi_-$ is thus equivalent to reducing the action (10) to

$$S_v \rightarrow \int d^2\vec{r}\, \frac{(\nabla \times \vec{a})_\tau^2}{2\langle y \rangle} \quad (13)$$

in the partition function $Z$. In the dielectric phase of the vortex system the field $\Phi_+$ is massless, and consequently $\langle y \rangle = 0$ [11], so the gauge field asymptotically decouples from the fermions. Quasiparticles become sharp excitations in the dSC, in agreement with the ARPES [6] and the microwave measurements [5]. In the nonsuperconducting phase, on the other hand, vortices are free, $\Phi_+$ becomes massive, and $\langle y \rangle \neq 0$. This has profound consequences for the fermions, as I discuss shortly.

At $T = 0$ quantum fluctuations need to be included, as the topological defects in (2 + 1) dimensions become vortex loops [12]. It seems clear on physical grounds, however, that after the integration over the loops (apart from the inherent anisotropy), the form of the action for the gauge field should remain similar to Eq. (13), except that $(\nabla \times \vec{a})_\tau^2/(2\langle y \rangle) \rightarrow (\nabla \times \vec{a})^2/(2\langle y \rangle)$. This can also be derived on a lattice, where one finds that the role of the coupling $\langle y \rangle$ at $T = 0$ is assumed by the *dual* order parameter that becomes finite only when there are infinitely large loops in the system and which is tantamount to the loss of phase coherence [13]. This way one finally arrives at the QED3 with the full Maxwell term for the transverse gauge field $\vec{a}$ as the relevant low-energy theory.

QED3 has been extensively studied by field theorists as a nontrivial toy model exhibiting the phenomena of dynamical symmetry breaking and confinement [4]. In particular, it has been established that for the number of Dirac fields $F < F_c$ the interaction with the gauge field is strong enough to spontaneously generate the so-called chiral mass for fermions. I will demonstrate that the chiral mass in the theory (11) is nothing but the SDW order parameter. First, to acquire some sense for the chiral instability, consider the fermion propagator. Neglecting the vertex and the wave-function renormalizations, it can be written as $G^{-1}(p) = i\gamma_\nu p_\nu + \Sigma(p)$, where the self-energy satisfies the self-consistent equation

$$\Sigma(q) = \langle y \rangle \gamma_\mu \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{D_{\mu\nu}(\vec{p} - \vec{q}) \Sigma(p)}{p^2 + \Sigma^2(p)} \gamma_\nu , \quad (14)$$

with $\vec{q} = (\omega, q_x, q_y)$. The gauge-field propagator in the transverse gauge is $D_{\mu\nu} = (\delta_{\mu\nu} - \hat{p}_\mu \hat{p}_\nu)/[p^2 + \Pi(p)]$, where $\Pi(p)$ is the self-consistently determined polarization. Assuming $m = \Sigma(0) \neq 0$ gives [4]

$$\Pi(q) = \frac{\langle y \rangle F}{2\pi} \left( m + \frac{q^2 - 4m^2}{2q} \sin^{-1} \frac{q}{\sqrt{q^2 + 4m^2}} \right). \quad (15)$$

When this is inserted into Eq. (14), it can be shown that there is a solution with a finite $m$ only when $F < F_c$, with $F_c = 32/\pi^2$ [4]. More elaborate calculations that fully include the wave-function renormalization and the vertex corrections confirm this result, and yield $F_c \approx 3$ [14]. Simulations on the lattice version of the QED3 [15] also find $3 < F_c < 4$, in agreement with the analytical estimates.

By reversing the transformations that led to the QED3 the reader can convince himself that the mass term for neutral fermions is equivalent to the low-energy part of the following addition to the electronic Hamiltonian (1):

$$mT \sum_{\vec{k}, \sigma, \omega_n, \vec{q} = \pm \vec{Q}_{1,2}} \sigma c_\sigma^\dagger(\vec{k} + \vec{q}, \omega_n) c_\sigma(\vec{k}, \omega_n), \quad (16)$$

so that the chiral mass may be identified with the SDW order parameter (or the staggered potential) along the spin $z$ axis, and at the wave vectors $\vec{Q}_{1,2}$. This is, of course, why the particular construction of the Dirac field was made in the first place. Note the following: (1) the SDW order is induced already at an infinitesimal vortex fugacity, but it is rather *weak,* $m \approx \langle y \rangle / \exp[2\pi/\sqrt{(F_c/F) - 1}]$ for $F \approx F_c$ [4], and (2) neutral fermions are bound (confined) at large distances in the SDW, by the weak logarithmic potential [provided by the fact that $\Pi(q) = \langle y \rangle F q^2/(6\pi m)$ for $q \ll m$]. With some anisotropy ($v_f \neq v_\Delta$), the global symmetry of the massless theory is only $U(2) \times U(2)$, so the mass term reduces each U(2) to $U(1) \times U(1)$. The two broken generators per Dirac field rotate the "cos-SDW" in Eq. (16) into either the similar "sin-SDW," or into the phase-incoherent state with "$d + ip$" pairing between the neutral fermions. SDW is therefore the only state that can be obtained by the unbinding of vortex defects and that respects parity. In the isotropic limit the massless theory recovers the full U(4) symmetry and additional broken symmetry states become available, like, remarkably, the stripelike charge density waves parallel to the $a$ or $b$ axis [13].

Once it is realized that unbinding of vortices leads to the SDW order, it becomes natural to wonder what the nature of vortices inside the dSC could be. From the perspective of this work it seems more than plausible that vortex cores are actually in the insulating phase, so that by approaching half-filling one lowers the core energy. In this picture the chemical potential should be related to the bare vortex fugacity, which when too large leads to the proliferation of defects, in analogy to the Berezinskii-Kosterlitz-Thouless transition [11]. The idea of SDW in vortex cores finds some experimental support in the recent STM [16] and the neutron scattering studies [17], as well as in the mean-field calculations [18]. Furthermore, the present work suggests that the superconductor-insulator transition should be accompanied by the appearance of the incommensurate SDW correlations at the wave vectors $\vec{Q}_{1,2}$, with the incommensurability *increasing* with doping. This is consistent with the recent neutron scattering experiments [19] on the underdoped LaSrCuO close to the superconducting transition. Finally, the $d$-wave pseudogap should continuously evolve into the insulating state, except for the gap that should develop at the nodes. This also seems in agreement with the observations [20].

To summarize, I showed how liberating topological defects in the $d$-wave superconductor at $T = 0$ leads to the incommensurate SDW, which is then expected to continuously evolve into the commensurate antiferromagnet close to half-filling. Near the transition the SDW order is inherently weak due to the relative closeness of the two flavor QED3 to its chiral critical point at $F_c \approx 3$. The SDW transition temperature near the superconductor-insulator transition may therefore be expected to be much lower than the corresponding superconducting $T_c$ on the other side of the transition, not in contradiction with the known topology of the cuprates phase diagram. The issue of a quantum-disordered (deconfined) ground state in this approach reduces to whether $F_c$ may, in fact, be smaller than two. It has recently been argued that $F_c \leq 3/2$ [21], in spite of the results of virtually every calculation violating this bound. Recent numerical study on larger systems than in [15], for example, find a small, but definitely a finite, mass at $F = 2$ [22]. A more interesting possibility is that $F_c$ may depend on some additional parameter in the theory. For example, one may speculate that a large anisotropy $v_f/v_\Delta \sim 10$, which is certainly present in cuprates, could affect $F_c$. This way $F_c$ would become doping dependent, which could open a route for the $T = 0$ deconfined phase in between the dSC and the SDW. Further examination of this possibility, together with the issue of the order of transition, effects of disorder and finite temperature, and the relations to other theoretical approaches to the high-$T_c$ problem will be discussed in the future longer publication [13].

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[1] J. Orenstein and A. J. Millis, Science **288**, 468 (2000).
[2] L. Balents, M. P. A. Fisher, and C. Nayak, Int. J. Mod. Phys. **10**, 1033 (1998); Phys. Rev. B **60**, 1654 (1999).
[3] M. Franz and Z. Tešanović, Phys. Rev. Lett. **87**, 257003 (2001); **84**, 554 (2000).
[4] T. Appelquist, D. Nash, and L. C. R. Wijewardhana, Phys. Rev. Lett. **60**, 2575 (1988).
[5] A. Hosseini *et al.,* Phys. Rev. B **60**, 1349 (1999).
[6] D. L. Feng *et al.,* Science **289**, 277 (2000).
[7] J. Corson *et al.,* Nature (London) **398**, 221 (1999).
[8] Z. A. Xu *et al.,* Nature (London) **406**, 486 (2000).
[9] See T. Senthil and M. P. A. Fisher, Phys. Rev. B **62**, 7850 (2000), for a different approach.
[10] See J. W. Negele and H. Orland, *Quantum Many-Particle Systems* (Addison-Wesley, Redwood City, CA, 1988), p. 204.
[11] J. M. Kosterlitz, J. Phys. C **7**, 1046 (1974); P. B. Weigmann, J. Phys. C **11**, 1583 (1978).
[12] A. Nguyen and A. Sudbo, Phys. Rev. B **60**, 15 307 (1999).
[13] I. F. Herbut (to be published). See also, Phys. Rev. Lett. **79**, 3502 (1997); Phys. Rev. B **57**, 13 729 (1998).
[14] P. Maris, Phys. Rev. D **54**, 4049 (1996).
[15] E. Dagotto, J. B. Kogut, and A. Kocić, Phys. Rev. Lett. **62**, 1083 (1989); Nucl. Phys. **B334**, 279 (1990).
[16] Ch. Renner *et al.,* Phys. Rev. Lett. **80**, 3606 (1998).
[17] B. Lake *et al.,* e-print cond-mat/0104026.
[18] J-X. Zhu and C. S. Ting, Phys. Rev. Lett. **87**, 147002 (2001).
[19] M. Fujita *et al.,* e-print cond-mat/0101320.
[20] F. Ronning *et al.,* Science **282**, 2067 (1998).
[21] T. W. Appelquist, A. G. Cohen, and M. Schmaltz, Phys. Rev. D **60**, 045003 (1999).
[22] J. Alexandre *et al.,* Phys. Rev. D **64**, 034502 (2001).