Full Frequency Back-Action Spectrum of a Single-Electron Transistor during Qubit Readout

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We calculate the spectral density of voltage fluctuations in a single-electron transistor (SET), biased to operate in a transport mode where tunneling events are correlated due to Coulomb interaction. The whole spectrum from low frequency shot noise to quantum noise at frequencies comparable to the SET charging energy ($E_C/\hbar$) is considered. We discuss the back-action during readout of a charge qubit and conclude that single-shot readout is possible using the radio-frequency SET.

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The single-electron transistor (SET) has recently been suggested as a readout device for solid-state charge qubits [1–4]. Presenting new state of the art figures for sensitivity in charge measurements, Aassime *et al.* [1] concluded that the radio-frequency SET [5] (RFSET) can be used for single-shot readout of the single-Cooper-pair box (SCB) qubit [6,7]. This is possible if the measurement time t_{ms} needed to resolve the two states of the qubit is much shorter than the time t_{mix} required to destroy the initial qubit state because of back-action due to voltage fluctuations on the SET.

The voltage fluctuations on the SET island have previously been studied in the experimentally accessible low frequency limit. In the classical transport regime of sequential tunneling the shot noise may be derived from a master equation with frequency independent tunneling rates [8–10], and recently the low frequency noise has also been calculated in the Coulomb blockade cotunneling regime [11,12].

In the low frequency limit the fluctuation energy $\hbar\omega$ is much smaller than the SET charging energy E_C, and thus the effect of the fluctuation energy $\hbar\omega$ on the SET tunneling rates may be neglected. In the high frequency limit ($E_C \ll \hbar\omega$) one may instead neglect both the driving bias of the SET and the quantization of its charge, resulting in the Nyquist noise from two independent unbiased tunnel junctions.

The qubit mixing rate $1/t_{\mathrm{mix}}$ is proportional to the noise at the frequency $\omega = \Delta E/\hbar$ corresponding to the energy splitting between the qubit states [13,14]. In Ref. [1] it was shown that back-action is minimized by maximizing ΔE during readout. For reasonable experimental parameters this gives that ΔE should be comparable to E_C. In order to estimate the back-action also in this relevant frequency regime ($E_C \sim \hbar\omega$) we evaluate the voltage fluctuations on the SET island on a fully quantum mechanical basis, taking into account the dependence of the tunneling rates on the fluctuation energy.

The main result of this paper is an expression for the voltage noise of a SET biased in the transport mode, valid in the whole frequency range from low frequency classical shot noise to high frequency quantum noise, separating processes that absorb/emit energy into negative/positive frequencies. This expression is then used to analyze the back-action of the SET on the SCB.

Consider a small metallic SET island coupled via low transparency tunnel barriers to two external leads, and coupled capacitively to a control gate and to the SCB (see Fig. 1). The voltage noise on the SET island is calculated while neglecting the coupling to the SCB. This approach is appropriate in the considered limit of weak SET-qubit coupling ($\kappa = C_c/C_{\mathrm{qb}}, \kappa \ll 1; C_L, C_R \sim C_{\mathrm{qb}}$).

We follow the outline of Ref. [15] and model the SET by the Hamiltonian

$$H = H_L + H_R + H_I + V + H_T = H_0 + H_T , \quad (1)$$

where

$$H_r = \sum_{\mathrm{kn}} \epsilon^r_{\mathrm{kn}} a^\dagger_{\mathrm{krn}} a_{\mathrm{krn}}, \qquad H_I = \sum_{\mathrm{ln}} \epsilon_{\mathrm{ln}} c^\dagger_{\mathrm{ln}} c_{\mathrm{ln}} \quad (2)$$

describe noninteracting electrons in the left/right lead ($H_r, r \in \{L, R\}$) and on the island (H_I). The quantum numbers n denote transverse channels including spin, and k, l denote momenta. The Coulomb interaction on the island is described by

$$V(\hat{N}) = E_C(\hat{N} - n_x)^2, \quad (3)$$

where $\hat{N}$ denotes the excess number operator, $E_C = e^2/2C$ the charging energy ($C = C_L + C_R + C_g + C_c$),

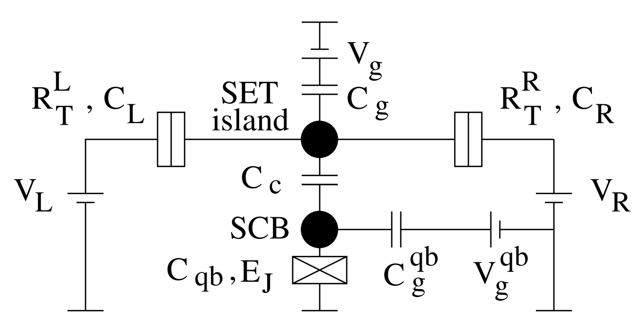

FIG. 1. Schematic figure of the SET capacitively coupled to the SCB.

n_x the fractional number of electrons induced by the external voltages [n_x is the fractional part of $(C_L V_L + C_R V_R + C_g V_g)/e$], and e the electron charge. The tunneling term is

$$H_T = \sum_{r=L,R} \sum_{\mathrm{kln}} (T^{\mathrm{rn}}_{\mathrm{kl}} a^\dagger_{\mathrm{krn}} c_{\mathrm{ln}} e^{-i\hat{\Phi}} + T^{\mathrm{rn}*}_{\mathrm{kl}} c^\dagger_{\mathrm{ln}} a_{\mathrm{krn}} e^{i\hat{\Phi}}), \quad (4)$$

where the operator $e^{\pm i\hat{\Phi}}$ changes the excess particle number on the island by ± 1 and $T^{\mathrm{rn}}_{\mathrm{kl}}$ are the tunneling matrix elements. $\hat{\Phi}$ is the canonical conjugate to $\hat{N}$, $([\hat{\Phi}, \hat{N}] = i)$. In this case of a metallic island containing a large number of electrons, the charge degree of freedom $N = 0, \pm 1, \ldots$ is to a very good approximation independent of the electron degrees of freedom l, n.

The spectral density of voltage fluctuations on the SET island is described by the Fourier transform of the voltage-voltage correlation function

$$S_V(\omega) = \frac{e^2}{C^2} \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} \mathrm{Tr}\{\rho_{\mathrm{st}}(t_0) \hat{N}(\tau) \hat{N}(0)\}. \quad (5)$$

Here $\rho_{\mathrm{st}}(t_0)$ is the density matrix of the system in steady state, which is assumed to have been reached at some time t_0 before the fluctuation occurs ($t_0 < \min\{0, \tau\}$). ρ_{st} is the tensor product of the equilibrium (Fermi distributed) density matrix ρ^e_{eq} for the electron degrees of freedom in each reservoir (L, R, I) and a reduced density matrix ρ^c_{st}, describing the charge degrees of freedom, which we assume to be diagonal [16] with elements P^{st}_N denoting the probability of being in charge state N.

To evaluate $S_V(\omega)$ we make a perturbation expansion of the forward and backward time evolution operators in terms of the tunneling term H_T. The reservoir degrees of freedom are traced out using Wick's theorem. In the diagrammatic language of Ref. [15] (Fig. 2) we have forward and backward propagators of the Keldysh contour (horizontal lines), with internal charge transfer vertices $e^{\pm\hat{\Phi}}$ (small dots) connected by reservoir lines. The $\hat{N}(0)$ and $\hat{N}(\tau)$ operators form external vertices (big dots). Figure 2 shows the diagrammatic expression for one term in $S_V(\omega)$ in our approximation. Here $\Pi_{NN'}(\omega)$ is the frequency dependent rate for transitions between the charge states N and N'. One may graphically write down a Dyson type of equation for $\Pi_{NN'}(\omega)$, in matrix notation this reads

$$\hat{\Pi}(\omega) = \hat{1}\frac{i}{\omega} + \hat{\Pi}(\omega)\frac{i\hat{\Sigma}(\omega)}{\omega} \Rightarrow \hat{\Pi}(\omega) = \frac{i}{\omega}\left[\hat{1} - \frac{i}{\omega}\hat{\Sigma}(\omega)\right]^{-1}, \quad (6)$$

where $\Sigma_{NN'}(\omega)$ represents the sum of irreducible diagrams, i.e., containing no free propagators, for transitions between N and N'.

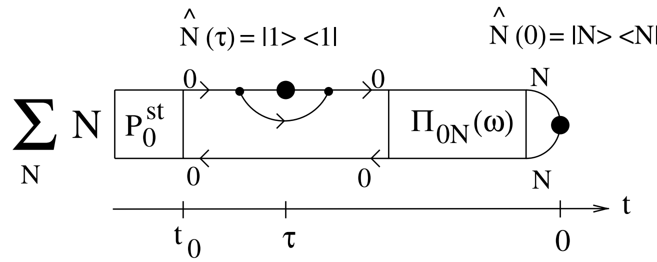

FIG. 2. The term $(i/\omega)P_0^{\mathrm{st}}\gamma_0^-(\omega)\Lambda_0$ in Eq. (11), in diagrammatic form.

Since the SET is biased in transport mode, we consider only the dominating lowest order diagrams in $\Sigma_{NN'}(\omega)$, corresponding to single-tunneling events. In this approximation we get [15]

$$\Sigma_{N,N\pm 1}(\omega) = \gamma^\pm_N(\omega) + \gamma^\pm_N(-\omega), \quad (7)$$

where $\gamma^\pm_N(\omega)$ denotes the rate, in second order pertubation theory, to go from charge state N to $N \pm 1$ while the SET is absorbing (ω) or emitting $(-\omega)$ a quantum of energy $|\hbar\omega|$. The well-known expressions for these rates are

$$\gamma^+_N(\omega) = \frac{\pi}{\hbar} \sum_r (\hbar\omega + \Delta^r_N) \alpha^r_0 n(\hbar\omega + \Delta^r_N), \quad (8)$$

$$\gamma^-_N(\omega) = \frac{\pi}{\hbar} \sum_r (\hbar\omega - \Delta^r_{N-1}) \alpha^r_0 n(\hbar\omega - \Delta^r_{N-1}), \quad (9)$$

where the dimensionless conductivity

$$\alpha^r_0 = \sum_n |T^{\mathrm{rn}}|^2 \rho^n_r \rho^n_I = \frac{R_K}{4\pi^2 R^r_T} \quad (10)$$

is the ratio of the quantum resistance $R_K = h/e^2$ and the resistance of barrier $r \in \{L, R\}$, and $\beta = 1/k_B T$. $n(E) = 1/\{1 - \exp[-\beta E]\}$ is the Bose-Einstein distribution of electron-hole excitations and comes from the convolution of the Fermi distributions for filled initial states and empty final states. $\Delta^r_N = V(N) + \mu_r - V(N+1) - \mu_I$ is the energy gained by an electron tunneling from the chemical potential of lead r (μ_r) to the chemical potential of the island (μ_I), thus changing the charge state from N to $N + 1$ (see Figs. 3a–3c). We also assume that the relaxation processes on the island are fast on the time scales we are looking at and neglect the energy dependence of both tunneling matrix elements $T^{\mathrm{rn}}_{\mathrm{kl}} \approx T^{\mathrm{rn}}$ and reservoir densities of states $\rho^n_r(\epsilon), \rho^n_I(\epsilon)$. To avoid renormalization effects [15] we choose not too small bias, i.e., $\max_r \alpha^r_0 \ln(E_c/|\Delta^r_0|) \ll 1$.

We also choose the bias not too high, so that the higher charge states are inaccessible at low frequency, i.e., $\gamma^+_1(0) = \gamma^-_0(0) = 0$ giving $P^{\mathrm{st}}_N = 0$ for $N \notin \{0, 1\}$. Fluctuations may then access $N \in \{-1, 0, 1, 2\}$ to leading order in the tunnel conductance, implying that $\hat{\Pi}(\omega)$ and $\hat{\Sigma}(\omega)$ in Eq. (6) are 4×4 matrices. The diagrammatic method then gives

$$S_V(\omega) = \frac{2e^2}{C^2}\,\mathrm{Re}\Big\{P_1^{\mathrm{st}}\Lambda_1 - \frac{i}{\omega}\big[(P_1^{\mathrm{st}}\bar{\gamma}_1^- - P_0^{\mathrm{st}}\gamma_0^+)(\Lambda_0 - \Lambda_1) + P_0^{\mathrm{st}}\gamma_0^-(\Lambda_0 - \Lambda_{-1}) + P_1^{\mathrm{st}}(\Sigma_{12} + \gamma_1^+)(\Lambda_2 - \Lambda_1)\big]\Big\}, \tag{11}$$

where $\Lambda_N = \sum_{N'} N'\Pi_{NN'}$, $\bar{\gamma} = \gamma(-\omega)$, and the frequency dependence has been suppressed for brevity. For arbitrary frequencies we evaluate Eqs. (6) and (11) numerically, but we may also get simple analytical results in two regimes.

At low frequencies $|\hbar\omega| \lesssim \min\{|\Delta_{-1}^{L/R}|, |\Delta_1^{L/R}|\}$ we may restrict ourselves to the two lowest charge states $N \in \{0,1\}$. The matrix inverse is then trivial, giving

$$S_V(\omega) = \frac{2e^2}{C^2}\,\frac{P_0^{\mathrm{st}}\gamma_0^+(\omega) + P_1^{\mathrm{st}}\gamma_1^-(\omega)}{\omega^2 + [\gamma_0^+(\omega) + \gamma_1^-(\omega) + \gamma_0^+(-\omega) + \gamma_1^-(-\omega)]^2}, \tag{12}$$

where $P_0^{\mathrm{st}} = \gamma_1^-(0)/[\gamma_0^+(0) + \gamma_1^-(0)]$ and $P_1^{\mathrm{st}} = 1 - P_0^{\mathrm{st}}$. At zero frequency this equation coincides with the classical shot noise result [8,10,14].

For high enough frequencies the fluctuation energy dominates over the biasing energy, i.e., $\Sigma_{NN'} \approx 2\pi\omega\alpha_0 \ll \omega$, and we may expand Eq. (6) to first order in $\hat{\Sigma}/\omega$, giving

$$S_V(\omega) = \frac{2e^2}{C^2}\,\frac{P_0^{\mathrm{st}}[\gamma_0^+(\omega) + \gamma_0^-(\omega)] + P_1^{\mathrm{st}}[\gamma_1^-(\omega) + \gamma_1^+(\omega)]}{\omega^2}. \tag{13}$$

In the high-frequency limit this expression approaches the Nyquist noise from the two unbiased junctions connected in parallel to ground.

Note the simple structure of Eqs. (12) and (13): a sum over the probabilities of being in the states $N = 0$ and $N = 1$ times the rates for possible transitions from these states, normalized by a denominator containing the finite lifetime of the states due to tunneling and the duration of the fluctuation. For arbitrary frequencies there is a similar structure, although the normalization is more complicated.

It is informative to analyze the noise at zero temperature where the tunneling rates are linear functions of the possible energy gain [since $n(E) = \Theta(E)$, where $\Theta(x)$ is the unit step function]. To be specific, we assume that $\mu_L > \mu_I > \mu_R$ and $\Delta_0 > 0$ so that electrons tunnel from the left lead to the right and $N = 0$ is the lowest charge state. We also assume that the left bias is larger than the right one, $|\Delta_0^L| > |\Delta_0^R|$ (see Fig. 3a).

For large negative frequencies $\hbar\omega < -|\Delta_0^L|$ the fluctuations try to extract more energy than the bias can provide, thus there is no contribution to the noise from this frequency range (since we neglect higher order cotunneling effects). In the range $-|\Delta_0^L| < \hbar\omega < -|\Delta_0^R|$ the SET may emit energy $|\hbar\omega|$ while tunneling through the left barrier, while for $-|\Delta_0^R| < \hbar\omega < 0$ tunneling through both barriers contributes. For positive frequencies the SET may absorb energy $\hbar\omega$ while tunneling through either barrier. For frequencies $\hbar\omega > |\Delta_0^{L/R}|$ there is also a contribution from electrons tunneling backwards, against the bias, at the (L/R) junction. In the region $|\Delta_0^{L/R}| \ll \hbar\omega < \{|\Delta_{-1}^{L/R}|, |\Delta_1^{L/R}|\}$ the higher charge states $N \in \{-1, 2\}$ are still inaccessible and the noise approaches exactly one half of the Nyquist noise. This may be explained by the strong correlation of fluctuations at the two barriers induced by the Coulomb interaction. At high frequencies $\hbar\omega > \{|\Delta_{-1}^{L/R}|, |\Delta_1^{L/R}|\}$ also processes at barrier L/R involving the charge states $N \in \{-1, 2\}$ contribute, giving the real high-frequency limit of Nyquist noise from two completely uncorrelated tunnel junctions. Figure 4 illustrates the above discussion of which processes contribute in which frequency region.

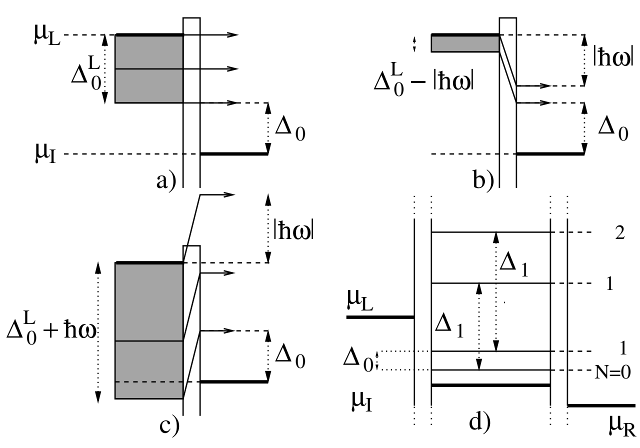

FIG. 3. (a)–(c) Schematic illustration of the frequency dependence of the rate for tunneling from the left lead to the island, when the island is in the $N = 0$ state. The shaded area indicates the amount of electrons energetically allowed to tunnel; (a) $\hbar\omega = 0$, (b) $\hbar\omega < 0$ (SET emits energy), and (c) $\hbar\omega > 0$ (SET absorbs energy). (d) Schematic figure showing the bias used in the discussion of the noise at different frequencies.

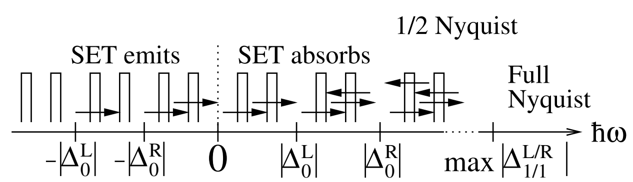

FIG. 4. Schematic picture of the processes contributing to the noise in different frequency regimes. Each arrow (passage through a barrier) involves emission or absorption of $|\hbar\omega|$.

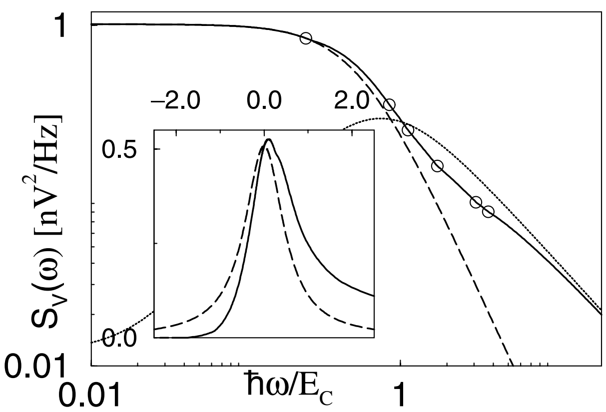

FIG. 5. Summed voltage noise $S_V(\omega) + S_V(-\omega)$ for a SET with parameters $R_t^L = R_t^R = 22$ kΩ, $n_x = 0.25$, $T = 20$ mK, $E_C/k_B = 2.5$ K, biased to dc current 6.7 nA. Full expression (solid line), classical shot noise (dashed line), Nyquist noise (dotted line). The circles denote, in order from left to right, $\hbar\omega = \{|\Delta_{-1}^R|, |\Delta_1^L|, |\Delta_0^L|, |\Delta_0^R|, |\Delta_{-1}^L|, |\Delta_1^R|\}$. The inset shows a comparison between the full expression (solid line) and classical shot noise (dashed line) in the low frequency regime (note the linear scale).

In the low-frequency regime $|\hbar\omega| < |\Delta_0^R|$, where no backward processes are allowed, the noise spectral density may be written as

$$S_V(\omega) = \frac{e^2}{C^2}\frac{2I/e + 2\pi\omega[P_0^{\rm st}\alpha_0^L + P_1^{\rm st}\alpha_0^R]}{\omega^2 + 4[\gamma_0^+(0) + \gamma_1^-(0)]^2}, \quad (14)$$

where $I = 2e\gamma_0^+(0)\gamma_1^-(0)/[\gamma_0^+(0) + \gamma_1^-(0)]$ is the dc current through the SET. This is the expression for classical shot noise plus a quantum correction term in the numerator, linear in ω, indicating that the probability of processes where the SET absorbs/emits energy ($\pm\omega$) is raised/lowered compared to the classical expression (see Figs. 3c, 3d and the inset of Fig. 5).

During the measurement, the voltage fluctuations on the SET island will induce transitions between the qubit states. The rate for relaxation/excitation of the qubit is [13,14]

$$\Gamma_{\rm rel/exc} = \frac{e^2}{\hbar^2}\kappa^2\frac{E_J^2}{\Delta E^2}S_V(\pm\Delta E/\hbar), \quad (15)$$

where E_J is the Josephson energy of the qubit. Since both relaxation and excitation corrupt the initial state, the mixing rate is the combined rate $1/t_{\rm mix} = \Gamma_1 = (\Gamma_{\rm rel} + \Gamma_{\rm exc})$. The sum $S_V(\omega) + S_V(-\omega)$, proportional to the back-action on the qubit, is plotted in Fig. 5 using the same junction parameters as Aassime *et al.* [1], biasing it to similar dc current within our normal state model. We find that simply adding the shot-noise and Nyquist expressions is a conservative estimate and single-shot readout is indeed possible. The slightly corrected value for the signal-to-noise ratio (SNR $= \sqrt{t_{\rm mix}/t_{\rm ms}}$), is now SNR $=$ 5 compared to 4. This small correction may be qualitatively understood as the shot-noise part being well approximated while the Nyquist expression overestimates the noise, since it neglects correlations between the junctions.

Note that the quantum correction becomes important at significantly lower frequencies for quantities that depend on the difference in relaxation and excitation rates. The inset of Fig. 5 shows how negative frequency noise, proportional to $\Gamma_{\rm exc}(\Delta E)$, falls to zero rather quickly on the scale of E_C, in contrast to the classical shot noise expression. It is clear that the relaxation of the qubit dominates over excitation at finite frequencies.

In conclusion, we have evaluated the finite frequency voltage fluctuations of a SET island biased in the transport mode. Using this information about back-action we conclude that single-shot readout of a SCB qubit using an RFSET is possible.

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