Experimental Evidence of Stochastic Resonance in an Excitable Optical System

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Experimental evidence of *stochastic resonance* in an excitable optical system is reported. We apply a sinusoidal forcing to the system and, for a finite external noise level, we find a frequency for which the excitable pulsing occurs periodically at the frequency imposed by the modulation. This resonant frequency matches the inverse of the average escape time of the stochastically driven system (i.e., without forcing). The same resonance is found by varying the noise level for fixed forcing frequencies. We discuss different indicators in order to describe quantitatively the degree of resonance.

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Introduction.—The notion of order in a nonlinear system is not trivially related to the amount of noise present in the system. Contrary to what happens in linear systems, where an increase of the noise amplitude leads to a degradation of the output signal, in nonlinear systems it has been shown that a finite amount of noise may induce a dynamical state which is, according to some indicator, more "ordered" [1–4]. Almost all real, nonisolated systems where the environment acts as a thermal bath are potential candidates to exhibit noise-ordered behavior, and indeed several cases of noise-induced ordering have recently been reported in several fields, ranging from biology to information theory and physics.

Two examples of noise-induced ordering in nonlinear dynamical systems are the so-called *stochastic resonance* (SR) and *coherence resonance* (CR). In both phenomena, the key concept is the interplay between a stochastic time scale, fixed by the noise level, D, and a deterministic time scale.

In bistable systems, an external modulation may lead to the phenomenon of stochastic resonance [2], where the noise-induced jumps between the two stable states of the system can be synchronized to the deterministic time scale externally imposed by the (weak) periodic forcing. This occurs when the noise level, D, is such that the corresponding Kramers' rate [5] equals half the period of the forcing. SR in an optical bistable system has been recently observed [4].

CR [6,7] occurs in excitable systems, which are characterized by their response to external perturbations: For perturbations that overcome a certain (nonzero) threshold, the system recovers its stable state (quiescent state) by emitting a pulse (excitable pulse) whose shape and duration are independent of the perturbation itself. The excitable pulse is associated with an orbit in phase space that requires a deterministic time T_r (the refractory time) to be completed [8]. In the presence of noise, the output of an excitable system consists of a train of noise-triggered excitable pulses with a temporal separation $T_p = T_r + T_a$. T_a is the stochastic activation time whose distribution depends on the noise level D through Kramers' law [5,9]. Therefore, as D increases, the average value of T_a decreases and $T_p \to T_r$; however, increasing noise levels will eventually distort the orbit associated with the excitable pulses. CR occurs for an optimal noise level, D_{CR}, low enough for the orbit being almost unaffected by noise but such that the system has high probability of escaping from the quiescent state as soon as it comes back: Then, the pulse train in the system output shows a maximum of regularity.

Excitable systems appear in many fields of science [10], including chemistry (Belousov-Zabotinskii reaction), biology (cardiac tissue and neurons) and physics (liquid crystals, optical systems). Hence, their response to external modulations and the nontrivial role of noise in these systems is of wide concern. In a recent paper, Wiesenfeld *et al.* [11] show that SR is also observable in excitable systems if the internal time scale determined by the refractory time is much smaller than the period of the external modulation. Their proposal relies on the fact that, when the Kramers' rate is much larger than the refractory time, the deterministic orbit plays a marginal role in determining the pulse time distribution. In this condition, an external forcing will compare directly to the stochastic time scale of the departures, and therefore the situation is very similar to the bistable SR except for having just a single stable state. As a consequence, the matching between the Kramers' rate T_k and the external modulation period T will be obtained when $T = T_k$ rather than $T = 2T_k$. It is worth noting that, for this interpretation to make sense, the modulation amplitude has to be always kept below the excitability threshold, while the modulation frequency has to be slow as compared to T_r, the intrinsic time scale in the system.

In this paper we give the first experimental demonstration of SR in an excitable optical system. The excitable system is a semiconductor laser with optical feedback, whose excitable characteristics have been recently demonstrated [12]. We add to the dc pump current a signal containing a broadband ($\Delta f \sim 1.6$ GHz), amplitude-controllable ac-coupled noise and a periodic modulation. By changing the noise level for fixed modulation frequency or, vice versa, changing the modulation frequency for a fixed noise level, we clearly demonstrate the existence of SR in the system. We discuss different indicators in order to precisely characterize the occurrence of SR.

The experimental setup.—The experimental setup is similar to the one described in [12]. An edge-emitting laser ($\lambda \sim 840$ nm) experiences optical feedback from a mirror. The external cavity length is 51 cm, corresponding to a free spectral range of 294 MHz. Such a system displays excitable behavior in a wide region in parameter space [12]. Within this region, the excitability threshold is set by the parameter values: It decreases when the current increases, while it increases when the feedback level increases. In order to observe SR in the clearest way, we have worked with the highest excitability threshold attainable in our experiment, though this choice is not critical for the observation of SR. The results presented in this paper have been obtained for a feedback level that induces a threshold reduction of 22% and a current value of 1% above the solitary laser threshold ($I_{\rm th} = 36.4$ mA). The laser output is monitored with an avalanche photodiode (2 GHz bandwidth) and sampled at a rate of 2 ns with a digital oscilloscope (500 MHz bandwidth). Through a bias-T, we superimpose to the dc bias current of the laser a periodic sinusoidal forcing and an amplitude-controlled, broadband ($\Delta f \sim 1.6$ GHz), zero-mean noise. The pulse statistics are computed from time series lasting for 4 ms which contain $(5-10) \times 10^3$ pulses.

Results and discussion.—For the above working conditions, without any external modulation the system is stable for low levels of noise. As the noise level is increased, the system starts to exhibit excitable pulses in the output. The histogram of T_p follows Kramers' law displaced by the refractory time, i.e., it displays no events for $T_p < T_r$ and from there on it is an exponentially decaying function of the time. As D increases, the Kramers' time, T_K, decreases until, at $D = D_{\rm CR}$, the system departs as soon as it returns to the quiescent state. At this noise level, we have a maximum of regularity in the pulse train, and the histogram of the times between pulses shows a narrow peak centered at T_r. Further increase of the noise deforms the refractory orbit, i.e., the pulse shape, and the coherence is lost. This phenomenon is the so-called CR [7].

In order to observe SR, we must work in the limit where the pulse duration is very short as compared to the typical time between pulses ($T_r \ll T_K$). In order to achieve this condition, the noise level has to be low, $D \ll D_{\rm CR}$, and it is convenient, though not essential, to have large excitability thresholds. This additional condition guarantees that $T_r \ll T_K$ over a large range of noise levels in spite of the high sensitivity of T_K to D. In this parameter range, we apply a periodic modulation of frequency ν and fixed amplitude (smaller than the excitability threshold) to the current. This is equivalent to modulating the excitablility threshold or, in other words, the probability of the system emitting an excitable pulse under the action of noise. If we

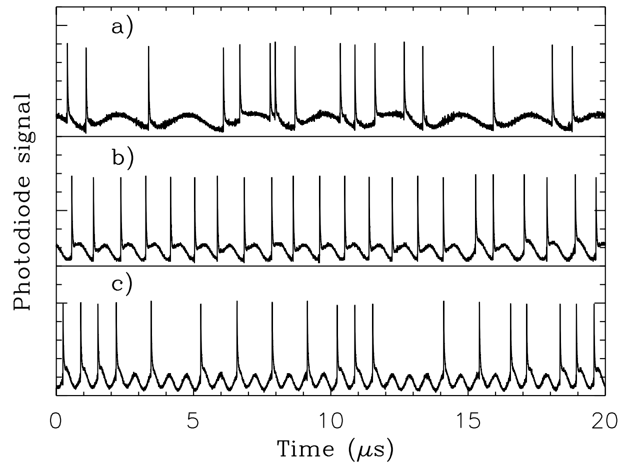

FIG. 1. Time traces of the laser output for a fixed noise level (-60.8 dBm V MHz$^{-1/2}$) and forcing frequency 0.4 MHz (a), 1.1 MHz (b), and 1.8 MHz (c). The vertical scale is the same for the three plots.

fix the noise level and we change ν, we are able to identify an optimal frequency $\nu_{\rm opt}$ for which the output of the system exhibits a highly regular train of pulses with T_p very close to the period of the forcing $T = \nu^{-1}$ (see Fig. 1b). For higher values of the forcing frequency $\nu > \nu_{\rm opt}$, the system does not depart at each period of the forcing and the pulse train loses regularity (Fig. 1c). For forcing frequencies $\nu < \nu_{\rm opt}$, the system may escape several times in the same forcing period (Fig. 1a), and again the pulse train is less regular than in Fig. 1b. If we change the noise level, we observe the same sequence of regimes, but with a different value of $\nu_{\rm opt}$.

In Fig. 2 we show, for different noise levels in the system, the probability distribution of T_p as the forcing frequency is varied. Several points are worth noting.

1. In all cases, there is a range of values of ν where one pulse is emitted at every period of the modulation. This range depends on the noise level (in Fig. 2 it corresponds to $600 < \nu < 1200$ KHz), and it shifts to higher values as the noise level is increased.

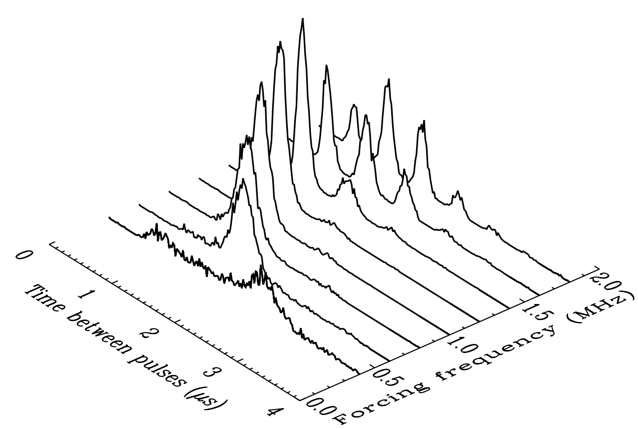

FIG. 2. Probability distribution of T_p for a fixed noise level (-65.5 dBm V MHz$^{-1/2}$) as the forcing frequency is varied.

2. For ν above this interval, the distribution of T_p exhibits multimodality, with peaks at integer multiples of the forcing period. This is due to the fact that, as shown in Fig. 1c, even if the system tends to depart at the peak of the sinusoidal modulation, it is not able to depart at every period. This effect may eventually lead to a distribution of T_p peaked around $2\nu^{-1}$ as in Fig. 2.

3. For ν above this range, T_p tends to be stochastically distributed, and the emission of excitable pules is no longer linked to the forcing because multiple pulses can be emitted during a modulation period.

The same sequence of behaviors can be observed by fixing ν and varying the noise level D. In Fig. 3 we show, for fixed forcing frequency, the distribution of T_p as D changes. For low noise levels, the distribution of T_p exhibits multimodality. As D is increased, the pulse firing tends to occur more and more in correspondence with the modulation period, until for large noise levels the system departs from the quiescent state several times within a modulation period.

In order to quantify the regularity of a given time series, an indicator of the degree of ordering of the pulses is required. Here, we search for an indicator that expresses as resonant a situation where both (i) one pulse is produced at every forcing period and (ii) the time intervals between the pulses are approximately always the same. Condition (i) does not imply a maximum of regularity since the pulsing may occur one time at each period but with a certain spread of phase, while condition (ii) concerns the spread of interpulse times but it does not imply that the system responds every period of the forcing.

Pikovsky's indicator for CR, $R = \sigma_p/\langle T_p\rangle$—where $\langle T_p\rangle$ is the mean value of the interpulse times and $\sigma_p = \langle (T_p - \langle T_p\rangle)^2\rangle^{1/2}$ is their standard deviation around the mean [6]—clearly accounts for the regularity of the interpulse times, but it does not consider that the degree of order should be measured in relation with the external forcing rather than the time series itself. For example, if our system responds with a pulse every two periods of the forcing, this will result in a very ordered time series with a zero value for the indicator R, but obviously this situation does not correspond to a 1:1 resonance with the external forcing. This restricts its applicability to the region of a 1:1 response, which has to be independently determined by some other method.

Since SR will occur when the pulsing frequency equals the frequency of the forcing, one possibility is to define the indicator from the power spectrum of the pulse train as in [13]. Another possibility is to use the indicator proposed in [4] for SR in bistable systems, which measures the area under a given peak in the histogram of residence times after subtraction of the background. However, this indicator requires an involved fitting procedure to separate the contribution of the background from that of the modulation.

Here we propose to measure the occurrence of SR through the indicator

$$I = \frac{1}{\sigma_p/T}\int_{(1-\alpha)T}^{(1+\alpha)T} dT_p\, f(T_p)\,, \tag{1}$$

where $f(T_p)$ is the distribution of interpulse times and $0 < \alpha < 0.25$ is a free parameter. This indicator combines the two desired features in a compact way: Its numerator evaluates the probability of the interpulse times to fall in a window of size 2α centered around the forcing period T, i.e., the fraction of pulses emitted with a separation roughly equal to the modulation period; its denominator is the standard deviation of T_p normalized to the modulation period, i.e., it accounts for the jitter between pulses. It is worth noting that the numerator is maximum within the regime of a 1:1 response, and in this regime the denominator is very close to Pikovsky's R. In the case of multipeaked histograms, the denominator is larger and the numerator is lower, thus leading to a reduction in I.

In Fig. 4 we plot I vs ν for different noise levels and $\alpha = 0.1$. For each noise level, I shows a clear maximum at an optimal frequency $\nu_{\rm opt}$, which is close to the center

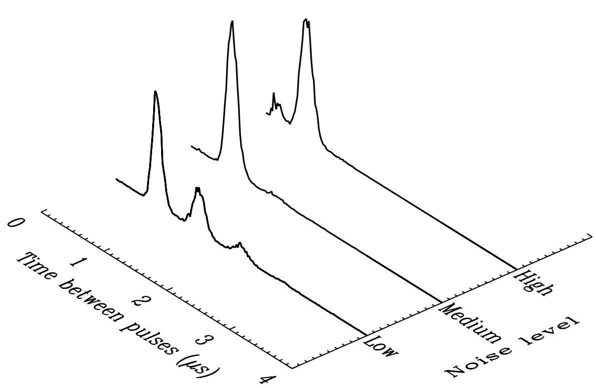

FIG. 3. Probability distribution of T_p, for $\nu = 1.4$ MHz for low -64.1 dBm V MHz$^{-1/2}$, medium (-57.9 dBm V MHz$^{-1/2}$), and high -51.0 dBm V MHz$^{-1/2}$ noise levels.

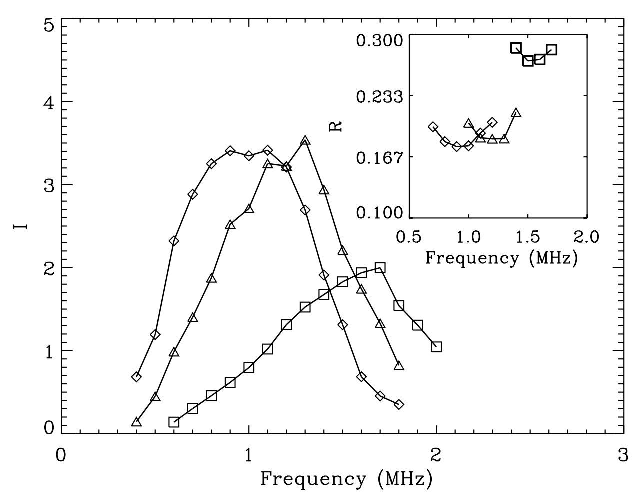

FIG. 4. I vs ν for noise levels -64.1 dBm V MHz$^{-1/2}$ ($\diamond$), -55.1 dBm V MHz$^{-1/2}$ ($\triangle$), and -51.0 dBm V MHz$^{-1/2}$ ($\square$), for $\alpha = 0.1$. The inset shows R vs ν for the same noise levels in the 1:1 regime.

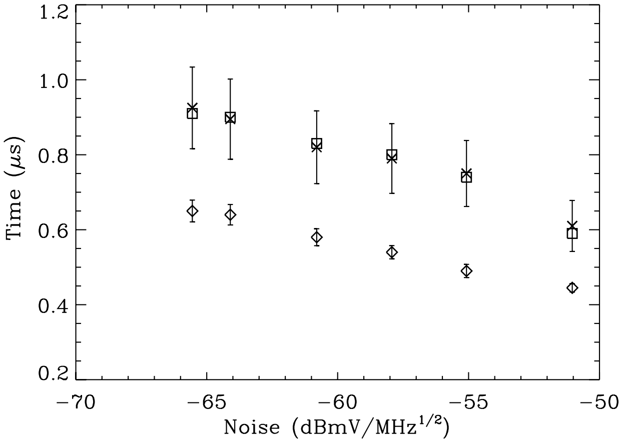

FIG. 5. Optimal modulation period, $\nu_{\mathrm{opt}}^{-1}(\square)$, as a function of noise level. Average interpulse time, $T_{\mathrm{ave}}(\times)$, and Kramers' time $T_K(\diamond)$ for the same noise levels in the system without modulation (see text for details).

of the frequency range where the system emits one pulse per modulation period (see Fig. 2). For the sake of comparison, the inset of Fig. 4 displays R in the 1:1 regime. It can be seen that ν_{opt} very closely corresponds to the frequency where a minimum in R is obtained, as expected from the above discussion. It is worth noting that choosing other values of α leads to an upward or downward shift of I, as should be expected because the area of integration under the peak in the histogram varies, but the resonance behavior and the optimal frequency do not change with α provided it is not taken too small or too large.

In Fig. 5 we plot, for the system without modulation, both the inverse of the Kramers' rate, T_K, and the average interpulse time, $T_{\mathrm{ave}} = \langle T_p \rangle$, as a function of D. T_K and its error bars have been determined by a fit of the histogram of interpulse times to Kramers' law. The error bars in $\langle T_p \rangle$ correspond to σ_p, and they have been directly computed from the interpulse time distribution. In order to establish a connection between the deterministic time scale imposed by the forcing period and the stochastic time scale intrinsic to the excitable system, we also plot the optimal modulation period, ν_{opt}^{-1}, for these same noise levels. It is evident that, in our experiment, ν_{opt}^{-1} decreases with D and that it coincides with the average escape time in the purely noise-driven system. This is a generalization of the result in [11], because in our experiment the distribution of escape times is not purely exponential, and in addition T_r is only about 1 order of magnitude smaller than T_{ave}; hence, this could make the approximation $T_K \approx T_{\mathrm{ave}}$ not strictly valid. Moreover, it has to be noted that the noise added to the system is not strictly white, and it has been recently shown [14] that this fact may quantitatively change the SR characteristics.

In conclusion, we have shown experimentally the existence of stochastic resonance in an optical excitable system. We characterized it by using a new statistical indicator that accounts for both the emission of one pulse per modulation period and the jitter of the pulse train. This indicator allows us to analyze the dependence of the optimal modulation frequency on the noise level, and its relation to the stochastic time scales in the purely noise-driven system.

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