Controlled Generation of Dark Solitons with Phase Imprinting

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The generation of dark solitons in Bose-Einstein condensates with phase imprinting is studied by mapping it into the classic problem of a damped driven pendulum. We provide a simple but powerful scheme, designing the phase imprint for various desired outcomes of soliton generation. For a given phase step, we derive a formula for the number of dark solitons traveling in each direction, and examine the physics behind the generation of counterpropagating dark solitons.

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Solitons have been discovered in various classical nonlinear media, such as fluid, magnetic, and optical systems, and have fascinated physicists for decades due to their particlelike properties [1]. Recently, dark solitons were observed in Bose-Einstein condensates (BECs) of dilute atomic gases [2,3], which are described by a macroscopic wave function. Dark solitons are produced by engineering the phase of this wave function with a technique known as phase imprinting, which was originally proposed and used to create vortices [4]. Phase imprinting is described as shining an off-resonance laser on a BEC in order to create phase steps between its different parts. As a new tool of manipulating matter waves, its power and limitation are yet to be fully understood.

In this Letter we present a thorough analysis on the generation of dark solitons with phase imprinting on BECs. Our study is greatly facilitated by a novel approach, which maps the soliton generation into the classic problem of a damped driven pendulum. We show how to design the phase steps for the desired number and characteristics of the dark solitons, and how to achieve such goals with minimum energy injection or disturbance to the system. Such controlled preparation of dark solitons is important for a detailed study of their interactions. We also derive a formula relating the winding number of the pendulum motion to the number of dark solitons traveling in each direction. In addition, we reveal the physics behind the production of counterpropagating dark solitons by one phase step. This rather mysterious phenomenon was observed in a recent experiment [2], where the situation is more complicated than our simple model. However, we believe that the physics should be the same. Although our study is done in the context of BEC physics, it can be readily applied to fiber optics, where dark solitons have potential applications in communication [5].

We consider a quasi-one-dimensional BEC, which is now realizable experimentally [6]. Also, because the generation of dark solitons has a very short time scale, we will neglect the trapping potential which mainly affects the subsequent dynamics after their generation [7]. Therefore, it is sufficient to use the one-dimensional nonlinear Schrödinger equation

$$i\,\frac{\partial\psi(x,t)}{\partial t} = -\frac{1}{2}\,\frac{\partial^2\psi(x,t)}{\partial x^2} + u_0^2|\psi(x,t)|^2\psi(x,t), \quad (1)$$

where x is measured in units of $\xi = 1\ \mu$m, a typical length unit in this type of experiments, t in units of $\frac{m\xi^2}{\hbar}$ (m is the atomic mass), ψ in units of the square root of n_0, the average density of the condensate, and $u_0 = \sqrt{4\pi n_0 a_s \xi^2}$ is the speed of sound, with $a_s > 0$ being the s-wave scattering length.

A dark soliton is characterized by a local density minimum moving with a constant speed against a uniform background of unit density [5,8]. It has three characteristics, the depth of its density minimum, the phase step over its density notch, and its velocity. All three are related to each other and can be specified by its velocity. The nonlinear Schrödinger equation is exactly solvable by the inverse scattering method [8], according to which the generation of dark solitons is determined by the Zakharov-Shabat (ZS) eigenvalue equations,

$$i\,\frac{\partial U_1(x)}{\partial x} + u_0\psi(x,0)U_2(x) = \lambda U_1(x), \quad (2)$$

$$i\,\frac{\partial U_2(x)}{\partial x} - u_0\psi^*(x,0)U_1(x) = -\lambda U_2(x), \quad (3)$$

where $\psi(x,0)$ is the initial wave function, such that $|\psi(x,0)| \to 1$ as $|x| \to \infty$. The ZS equations can have real discrete eigenvalues λ_i with magnitude smaller than u_0. Corresponding to each λ_i, a dark soliton with velocity $-\lambda_i$ will be generated. For phase imprinting, we have $\psi(x,0) = e^{iS(x)}$, where $S(x)$ is the imprinted phase. For simplicity and without loss of generality, we will limit our attention to the right phase steps, which increase from the left to the right and approach constants at $\pm\infty$.

We solve the ZS equations by mapping them into a simple pendulum problem, which is physically more intuitive and mathematically much simpler. For real eigenvalues, the quantity $|U_1|^2 - |U_2|^2$ and the overall phase of the two amplitudes are independent of position. For discrete eigenvalues, both U_1 and U_2 approach zero at spatial infinity. These results imply that the two amplitudes have equal magnitudes and may be chosen to have opposite phases

$$U_1 = i\sqrt{\rho}\, e^{i(-\varphi+S)/2}, \qquad U_2 = \sqrt{\rho}\, e^{i(\varphi-S)/2}. \tag{4}$$

This turns the ZS equations into a pair of very simple equations

$$\dot{\varphi} = 2\lambda + \dot{S} - 2u_0 \sin\varphi\,, \tag{5}$$

$$\dot{\rho} = 2u_0 \rho \cos\varphi\,, \tag{6}$$

where the overhead dot denotes the spatial derivative. Remarkably, Eq. (5) involves only φ, and can be viewed as a damped massless pendulum driven by the force $2\lambda + \dot{S}$ if we regard x as time. This analogy works even for an inhomogeneous initial density, in which case u_0 should be replaced by the local sound speed. We will, however, leave the discussion of density engineering [9] of dark solitons to a future publication.

In the asymptotic regime $x \to \pm\infty$, where the slope $\dot{S}$ of the imprinted phase turns to zero, Eq. (5) has two fixed points: $P_s(\lambda)$ at $\varphi_0 = \sin^{-1}\frac{\lambda}{u_0}$ and $P_u(\lambda)$ at $\pi - \varphi_0$ (see Fig. 1). Therefore, all pendulum motions, as governed by Eq. (5), always start at a fixed point and finish at a fixed point. Among the four possibilities, $P_s \to P_s$, $P_u \to P_u$, $P_u \to P_s$, and $P_s \to P_u$, only the last one corresponds to the ZS eigensolutions. All the other three types of pendulum motion yield, according to Eq. (6), a divergent ρ at spatial infinity, which violates the boundary condition for the ZS eigensolutions.

That the solution $P_s \to P_u$ is special can be appreciated from another angle by noticing that P_s is a stable fixed point while P_u is unstable. For a given phase S and a general value of λ, the pendulum starting at P_s almost always is pulled into the stable fixed point P_s eventually [see Fig. 1(a)]. Only for a discrete set of λ, the motion ends exactly at the unstable fixed point P_u, and stays there afterwards [see Fig. 1(b)]; these special λ values are just the eigenvalues λ_i of the ZS equations.

Our approach is novel and has certain advantages over the existing methods for the study of ZS equations [10]. From the above analysis it is clear that we can discard Eq. (6) and focus only on the pendulum equation (5), which is much simpler than the ZS equations. For example, we solve Eq. (5) for the case of the "sudden" limit in which the phase imprinted is a step function [11]. The corresponding force is a δ function, $\dot{S} = \Delta\delta(x)$, which imparts an angular change of the pendulum equal to the step height. This should make up the phase difference between the unstable and stable fixed points, $(\pi - \varphi_0) - \varphi_0 = \pi - 2\sin^{-1}\frac{\lambda}{u_0}$. So only one soliton can be generated, and its speed is $\lambda = u_0 \sin\varphi_0 = u_0 \cos(\Delta/2)$. Later, we will provide analytic solutions for another case, where the phase-step width is finite and multiple solitons can be generated.

More importantly, the simplicity of our approach allows us to ask and answer the inverse question, "What phase step is needed to produce a specified dark soliton?" We can design the phase imprint for dark-soliton generation with the following steps: (1) pick a λ, the soliton that one wishes to create; (2) choose a curve, $\dot{\varphi} = f(\varphi)$, connecting the pair of fixed points $P_s(\lambda)$ and $P_u(\lambda)$ in Fig. 2; (3) solve for $\varphi(x)$ from this curve, and substitute it into Eq. (5) to obtain $S(x)$. The obtained phase step $S(x)$ will create the soliton of velocity $-\lambda$.

Obviously, there is an infinite number of paths connecting a pair of P_s and P_u, so there is an infinite number of different phase steps that generate a certain dark soliton. We can optimize our design of the phase step, such as the minimization of energy injection into the system or equivalently the noise output accompanying the generation of the dark soliton. The energy injection by imprinting a phase step $S(x)$ is given by

$$E = \int_{-\infty}^{\infty} dx\, \frac{\dot{S}^2}{2} = \int_{\varphi_0}^{\pi-\varphi_0} d\varphi\, \frac{\dot{S}^2}{4\lambda - 4u_0 \sin\varphi + 2\dot{S}}\,. \tag{7}$$

This is a functional in $\dot{S}$. A variation over $\dot{S}$,

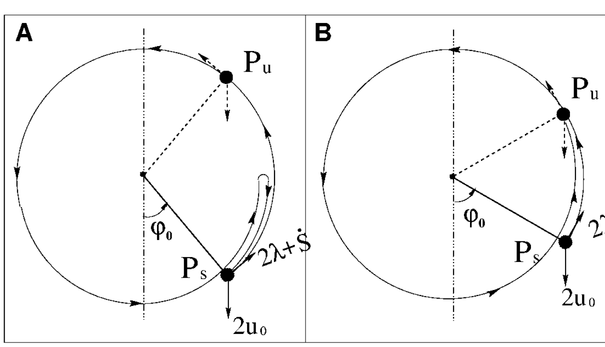

FIG. 1. Motions of pendulum (5). Trajectories are schematic, and deliberately distorted when the overlapping occurs. The vectors are forces. (a) Motion starting from P_s and coming back to P_s after one rotation. (b) Motion going from the stable fixed point P_s to the unstable fixed points P_u.

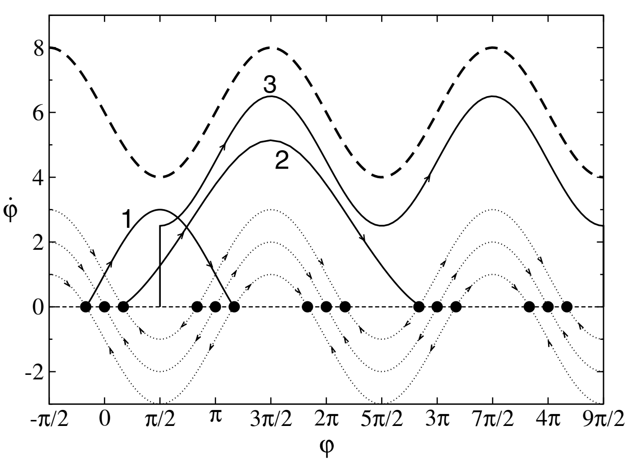

FIG. 2. Phase trajectories of a damped driven pendulum of zero mass. Dotted lines are drawn for constant forces 2λ; the solid lines are for changing forces $2\lambda + \dot{S}$. The arrows indicate the directions of pendulum motion. The full circles are the stable fixed points P_s; the open circles are the unstable fixed points P_u.

$$\frac{\delta}{\delta \dot{S}}\left(\frac{\dot{S}^2}{4\lambda - 4u_0 \sin\varphi + 2\dot{S}}\right) = 0, \tag{8}$$

combined with Eq. (5), yields the designing curve $\dot{\varphi} = -2\lambda + 2u_0 \sin\varphi$ (curve 1 in Fig. 2) and the optimum phase step

$$S(x) = 4\tan^{-1}\left(\frac{u_0 - \lambda}{\sqrt{u_0^2 - \lambda^2}} \tanh(x\sqrt{u_0^2 - \lambda^2})\right). \tag{9}$$

This phase step generates a soliton of velocity $-\lambda$, while producing the smallest possible disturbance to the system. This "best" phase step varies monotonically, and resembles the shape of the phase step used in experiments. However, the minimum energy injection,

$$E = 8u_0\left[\tan\left(\frac{\pi}{2} - \varphi_0\right) - \left(\frac{\pi}{2} - \varphi_0\right)\right]\sin\varphi_0, \tag{10}$$

still exceeds the energy of a pure dark soliton,

$$E_s = \tfrac{4}{3}\, u_0 \cos^3\varphi_0 \,. \tag{11}$$

This exposes the limitation of phase imprint: there are always some noise and possibly other solitons generated along with the desired dark soliton.

The number of dark solitons that can be generated by a phase step is related to the winding numbers of the pendulum motion in a simple way. Earlier we mentioned that for a general λ the pendulum starting at a stable fixed point $P_s(\lambda)$ comes back to either $P_s(\lambda)$ or $P_u(\lambda)$ after a number of complete rotations. We call the number of full rotations completed the winding number $W(\lambda)$. The winding number remains the same as λ varies between the eigenvalues and increases by one when it crosses each eigenvalue [12]. Therefore, the number of eigenvalues in the allowable range $(-u_0, u_0)$ is given by the simple formula $N_s = W(u_0) - W(-u_0)$, which yields the total number of dark solitons that can be generated by the phase step. Similarly, the number of dark solitons that moves to the right is given by $N_r = W(0) - W(-u_0)$, and the number moving to the left is given by $N_l = W(u_0) - W(0)$.

With this simple and interesting relation between the number of dark solitons and the winding number, we can design phase steps to produce exactly n dark solitons. This is achieved by finding a phase step that yields $W(u_0) = n$ and $W(-u_0) = 0$. Such a phase step $S(x)$ is obtained with Eq. (5) by drawing a path in Fig. 2 connecting $\varphi = \pi/2$ and $\varphi = 2n\pi + \pi/2$, the fixed points of the pendulum driven by the constant force $2u_0$. At the same time, we make sure that the path lies below the curve, $\dot{\varphi} = 6u_0 - 2u_0 \sin\varphi$, the darkened dashed line in Fig. 2 [13]. For example, curve 3 in Fig. 2, which represents a linear phase step to be discussed below, generates exactly two dark solitons.

Finally, we apply our results to a simple but very useful case, the linear phase step, whose slope is $\dot{S} = \alpha/a > 0$ for $|x| < a$ and zero elsewhere. The phase steps created in the present experiments [2,3] can be well modeled by this linear phase step. For this simple case Eq. (5) can be solved analytically, and the eigenvalues λ_i are given by the roots of

$$\frac{\lambda^2 - u_0^2 + \lambda\frac{\alpha}{2a}}{\sqrt{u_0^2 - \lambda^2}} = \frac{\sqrt{(\lambda + \frac{\alpha}{2a})^2 - u_0^2}}{\tan[a\sqrt{(\lambda + \frac{\alpha}{2a})^2 - u_0^2}]}. \tag{12}$$

The two winding numbers $W(u_0)$ and $W(-u_0)$ of the linear phase step can also be computed exactly, yielding the following simple formula for the number of dark solitons generated by the phase step

$$N_s = \mathrm{Int}\left(\frac{1}{\pi}\sqrt{\frac{\alpha^2}{4} + u_0\alpha a}\right) - \mathrm{Int}\left(\frac{1}{\pi}\sqrt{\frac{\alpha^2}{4} - u_0\alpha a}\right) + 1, \tag{13}$$

where Int takes the integer part of the real numbers, and the second term is to be omitted if $\alpha a < 4u_0$. With both Eqs. (12) and (13), we obtain the soliton velocities, and find the number of solitons generated. In Fig. 3, we plot the total number of solitons generated and the nmber of solitons traveling to the right. As a whole, Fig. 3 can serve as a reference table for experimentalists to find the right parameters, the step height and step width, to generate the desired number of solitons traveling in each direction. A figure of the same type is seen in Ref. [14].

The generation of left-moving dark solitons by a right phase step (the focus of our paper) can be understood intuitively. Once a right phase step ($\dot{S} \geq 0$) is imprinted on a BEC cloud, atoms in the step area will start moving to the right. Since the atoms outside of the step area do not move, a dip with a bump to its right will appear as a result (see Fig. 4). Because of the stronger repulsive interaction from the bump, the dip will be pushed to the left. As dark solitons come into form in the dip, one would expect that the dark solitons generated by a right phase step would always move to the left. Indeed, a comparison of

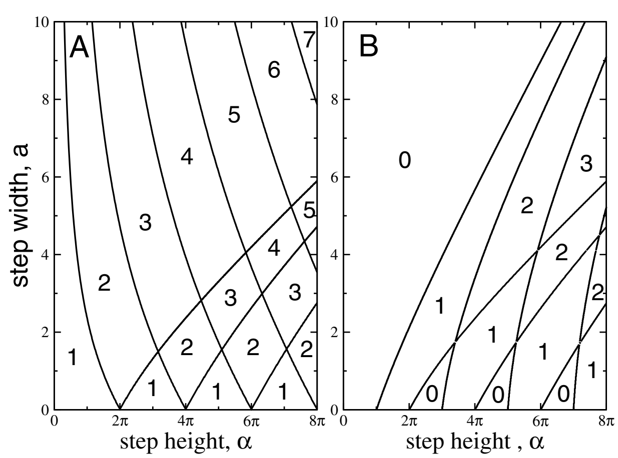

FIG. 3. (a) Number of solitons. (b) Number of solitons traveling to the right. The numbers, such as "0, 1, 2, . . . ," in the figure mark the number of solitons generated in the parameter regions defined by the solid lines.

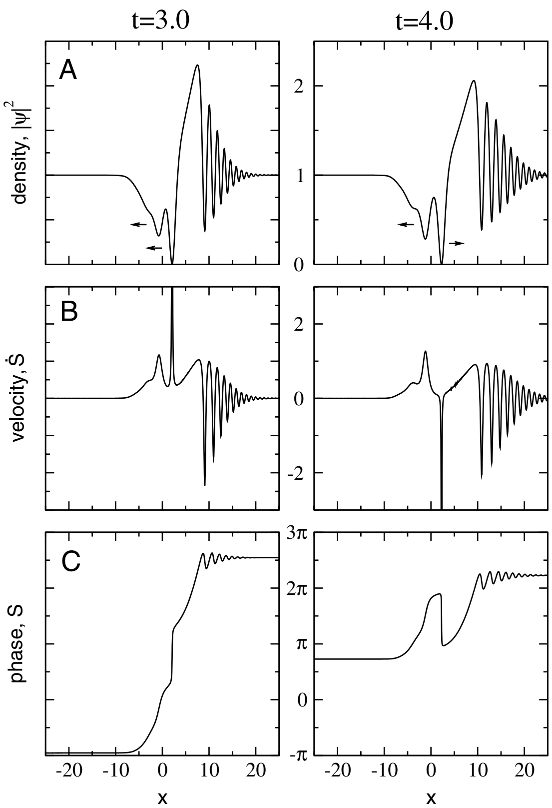

FIG. 4. Generation of counterpropagating dark solitons. Two snapshots are taken at $t = 3.0$ and $t = 4.0$. This initial phase step is a tanh function, $S(x) = \alpha[\tanh(2x/a) + 1]/2$, with $u_0 = 1.0$, $a = 3.6$, and $\alpha = 3.5\pi$.

Figs. 3(a) and 3(b) shows that it is much easier to generate left-moving dark solitons than the right-moving ones.

However, right-moving dark solitons do get generated with the correct choice of height and width of the right phase step, as indicated by the areas marked with nonzero numbers in Fig. 3(b). Such counterpropagating dark solitons are indeed observed experimentally [2]. Our diagram Fig. 3 shows that it is even possible to generate a single dark soliton that travels in the "wrong" direction. To understand this rather mysterious phenomenon, we plot the time evolution of the soliton generation in Fig. 4. The intuitive picture discussed in the previous paragraph is still correct at the initial stage (Fig. 4, $t = 3.0$), where the two dips do move to the left. A short time later ($t = 4.0$), the deeper dip reverses its direction and starts to move to the right. This is accompanied by a slippage of 2π in the phase step [Fig. 4(c)], which can take place when the density at the bottom of the dip reaches zero. In Ref. [15], a conservation law was established for the sum of the soliton momenta and the field momentum defined as the total phase difference $S(+\infty) - S(-\infty)$. Therefore, the velocity reversal of the deeper dip must be due to the exchange of momentum with the field.

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[13] Suppose the path is $\dot{\varphi} = f(\varphi) < 6u_0 - 2u_0 \sin\varphi$. Substituting this in Eq. (5) for $\lambda = u_0$ (the path is drawn for $\lambda = u_0$), we have an inequality, $0 \leq \dot{S} < 4u_0$. Therefore, for this phase step we always have $\dot{\varphi} < 0$ at $\varphi = \pi/2$ with $\lambda = -u_0$. This implies that in this case the pendulum can never pass $\varphi = \pi/2$ thus its winding number $W(-u_0) = 0$.
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